Lecture Notes: To Replace 7.6-7.8

Today we will finish computing the solution to $\mathbf{x}' = A\mathbf{x}$ by looking at the case of *complex eigenvalues* and *one real eigenvalue*.

Last time, we saw that, to compute eigenvalues and eigenvectors for a matrix A, we first compute the characteristic equation, then solve for a representative eigenvector.

We applied this to $\mathbf{x}' = A\mathbf{x}$ for "Case 1" which was when we had two distinct real eigenvalues, $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$, and saw that the general solution is:

$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

Case 2: Complex Eigenvalues

First, let's look at the eigenvalue/eigenvector computations themselves in an example: Find the eigenvalues and eigenvectors for the matrix below:

$$\begin{aligned} 3v_1 - 2v_2 &= \lambda v_1 \\ v_1 + v_2 &= \lambda v_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (3-\lambda)v_1 - 2v_2 &= 0 \\ v_1 + (1-\lambda)v_2 &= 0 \end{aligned}$$

SOLUTION: Form the characteristic equation using the shortcut or by taking the determinant of the coefficient matrix:

$$\lambda^2 - Tr(A)\lambda + \det(A) = 0 \qquad \lambda^2 - 4\lambda + 5 = 0 \qquad \lambda = 2 \pm i$$

Now, if $\lambda = 2 + i$, solve for an eigenvector:

$$\begin{array}{rrl} (3-(2+i))v_1 & -2v_2 & = 0 \\ v_1 & +(1-(2+i))v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} (1-i)v_1 & -2v_2 & = 0 \\ v_1 & +(-1-i)v_2 & = 0 \end{array}$$

Side Note/Side Computation

Recall that we said that these equations needed to be the same line- Indeed they are. To see this, if you divide the first equation by $1-i$, we get:

$$\frac{1-i}{1-i}v_1 - \frac{2}{1-i}v_2 = 0 \quad \Rightarrow \quad v_1 - \frac{2(1+i)}{(1^2+1^2)}v_2 = 0 \quad \Rightarrow v_1 - (1+i)v_2 = 0$$

which is the second equation.

Returning to the Problem...

Given $(1-i)v_1 - 2v_2 = 0$, we can use $\mathbf{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$.

As a side remark, the other eigenvalue/eigenvector pair are the complex conjugates (we won't be using them):

$$\lambda_2 = 2 - i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

The next section tells us how to solve the system.

Applying Complex evals to Systems of DEs

Suppose we have a complex eigenvalue, $\lambda = a \pm ib$. Use one of them to construct the corresponding eigenvector (complex) $\mathbf{v}$. We can then solve the system using the theorem below.

Theorem: Given $\lambda = a + ib$, $\mathbf{v}$ for a matrix A in $\mathbf{x}' = A\mathbf{x}$, the solution to the system of differential equations is:

$$\mathbf{x}(t) = C_1 \mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

Notice that this is the extension of what we did in Chapter 3.

Example

Give the general solution to the system $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \mathbf{x}$

This is the system for which we already have the eigenvalues and eigenvectors:

$$\lambda = 2 + i \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 1 - i \end{bmatrix}$$

Now, compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(2+i)t} \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} = \mathrm{e}^{2t}(\cos(t) + i\sin(t)) \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} =$$

$$\mathrm{e}^{2t} \begin{bmatrix} 2\cos(t) + 2i\sin(t) \\ (\cos(t) + \sin(t)) + i(-\cos(t) + \sin(t)) \end{bmatrix}$$

so that the general solution is given by:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix} + C_1 \mathrm{e}^{2t} \begin{bmatrix} 2\sin(t) \\ -\cos(t) + \sin(t) \end{bmatrix}$$

Geometrically, the origin is a *spiral source*. As a side remark, if I had solved the second equation for x_1 and substituted it into the first, I would have had:

$$x_2'' - 4x_2' + 5x_2 = 0 \quad \Rightarrow \quad r = 2 \pm i \quad \Rightarrow \quad x_2 = C_1 \mathrm{e}^{2t}\cos(t) + C_2 \mathrm{e}^{2t}\sin(t)$$

Example

Give the general solution to the system: $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

First, the characteristic equation: $\lambda^2 + 1 = 0$, so that $\lambda = \pm i$.

Now we solve for the eigenvector to $\lambda = i$:

$$\begin{aligned} (2 - i)v_1 - 5v_2 &= 0 \\ 1v_1 + (-2 - i)v_2 &= 0 \end{aligned}$$

Using the second equation, $v_1 - (2+i)v_2 = 0$, and we have our eigenvalue/eigenvector pair. Now we compute the needed quantity, $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{it}\left[\begin{array}{c} 2+i \\ 1 \end{array}\right] = (\cos(t) + i\sin(t))\left[\begin{array}{c} 2+i \\ 1 \end{array}\right] = \left[\begin{array}{c} (\cos(t)+i\sin(t))(2+i) \\ \cos(t)+i\sin(t) \end{array}\right]$$

Simplifying, we get:

$$\left[\begin{array}{c} (2\cos(t)-\sin(t)) + i(2\sin(t)+\cos(t)) \\ \cos(t)+i\sin(t) \end{array}\right]$$

The solution is:

$$\mathbf{x}(t) = C_1\left[\begin{array}{c} 2\cos(t)-\sin(t) \\ \cos(t) \end{array}\right] + C_2\left[\begin{array}{c} 2\sin(t)+\cos(t) \\ \sin(t) \end{array}\right]$$

We will quickly verify that this is what we would get using the techniques of Chapter 3. From the second equation, solve for x_1, then use the first equation to get a second order DE for x_2.

$$x_1 = x_2' + 2x_2 \quad \Rightarrow \quad (x_2'' + 2x_2') = 2(x_2' + 2x_2) - 5x_2 \quad \Rightarrow \quad x_2'' + x_2 = 0$$

Therefore, $x_2 = C_1\cos(t) + C_2\sin(t)$. Solving for x_1:

$$x_1 = x_2' + 2x_2 = (-C_1\sin(t) + C_2\cos(t)) + 2(C_1\cos(t) + C_2\sin(t))$$

and we see that we get the identical solution.

Graphically, the solutions are ellipses. In fact, if we solve the differential equation by computing dy/dx, we get solutions of the form:

$$x^2 - 4xy + 5y^2 = C$$

Graphical Summary- Complex Eigenvalues

Notice that if the real part of λ is positive, solutions "blow up". If the real part of λ is negative, $y(t) \to 0$ as $t \to \infty$. Therefore, the origin can be classified by $\lambda = \alpha \pm \beta i$:

- If $\alpha = 0$, we get pure periodic solutions (the period depends on β).
- If $\alpha < 0$, the origin is a *spiral sink.*
- If $\alpha > 0$, the origin is a *spiral source.*

Case 3: One Real Eigenvalue, One Eigenvector

In the rare occurrence that you have one eigenvalue but two eigenvectors go to Case 1. For example, find the eigenvalues and eigenvectors to the identity matrix.

$$\left| \begin{array}{cc} (1-\lambda) & 0 \\ 0 & (1-\lambda) \end{array} \right| = 0 \quad \Rightarrow \quad \lambda = 1, 1$$

Now, solve the system for $\mathbf{v}$:

$$\begin{aligned} 0v_1 + 0v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned}$$

Both v_1, v_2 are free variables, so any vectors would work- We could use any two vectors (non-zero, not multiples of each other) to be our eigenvectors. Some like to use:

$$\mathbf{v} = v_1 \left[\begin{array}{c} 1 \\ 0 \end{array} \right] + v_2 \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$$

And therefore, we have two eigenvectors, $[1, 0]^T$ and $[0, 1]^T$. This is not typical.

Typical Case: A double eigenvalue, one eigenvector

Example: $\left[\begin{array}{cc} 2 & 3 \\ 0 & 2 \end{array} \right]$ In this case, $\lambda = 2, 2$ but

$$\begin{aligned} 0v_1 + 3v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$$

This one can be a little tricky, but there is an way to quickly get the solution if we have the intial conditions, $\mathbf{x}(0) = \mathbf{x}_0$. Then the solution to $\mathbf{x}' = A\mathbf{x}$ is given by:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

If we substitute this back into the DE, we will see that the following needs to hold:

$$(A - \lambda I)\mathbf{x}_0 = \mathbf{w}$$

Example:

$$\mathbf{x}' = \left[\begin{array}{cc} 2 & 3 \\ 0 & 2 \end{array} \right] \mathbf{x}, \qquad \mathbf{x}_0 = \left[\begin{array}{c} x_0 \\ y_0 \end{array} \right]$$

We just computed the eigenvalues to be $\lambda = 2, 2$. To find the vector $\mathbf{w}$, we take:

$$\begin{aligned} (2-2)x_0 + 3y_0 &= w_1 \\ 0x_0 + (2-2)y_0 &= w_2 \end{aligned} \quad \Rightarrow \quad \mathbf{w} = \left[\begin{array}{r} 3y_0 \\ 0 \end{array} \right]$$

The full solution is then:

$$\mathbf{x}(t) = \mathrm{e}^{2t} \left(\left[\begin{array}{c} x_0 \\ y_0 \end{array} \right] + t \left[\begin{array}{r} 3y_0 \\ 0 \end{array} \right] \right)$$

Example:

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x}$$

The trace is 0 and the determinant is 0. Therefore, $\lambda = 0$ is the only eigenvalue. If there are no initial conditions, assume they are (x_0, y_0) as the last example. Then our vector $\mathbf{w}$ is computed as:

$$\begin{array}{ll} (4-0)x_0 - 2y_0 & = w_1 \\ 8x_0 - (4-0)y_0 & = w_2 \end{array} \qquad \mathbf{w} = \begin{bmatrix} 4x_0 - 2y_0 \\ 8x_0 - 4y_0 \end{bmatrix}$$

The solution is (in several forms):

$$\mathbf{x}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 4x_0 - 2y_0 \\ 8x_0 - 4y_0 \end{bmatrix}$$

We'll note that this is just a straight line in the (x_1, x_2) plane.

Summary

To solve $\mathbf{x}' = A\mathbf{x}$, find the trace, determinant and discriminant. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$ (not used directly).

  Use the initial condition, $\mathbf{x}_0 = (x_0, y_0)$ and the vector $\mathbf{w}$ so that

$$\begin{array}{ll} (a-\lambda)x_0 + by_0 & = w_1 \\ cx_0 + (d-\lambda)y_0 & = w_2 \end{array} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{x}_0 = \mathbf{w}$$

  The solution is then

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$