Exercise Set 3 (HW for 7.6, 7.8)

In this homework set, we will practice finding eigenvalues and eigenvectors when the eigenvalues are either complex or the matrix is defective.

1. You may skip this problem- I had meant to delete it.
2. Exercises 1, 3, pg. 409 (Section 7.6, solve with complex evals/evecs)

   SOLUTIONS: For exercise 1 in the book, the matrix is below.

   $$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \qquad \lambda^2 - 2\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda - 1)^2 = -4 \quad \Rightarrow \quad \lambda = 1 \pm 2i$$

   Using $\lambda = 1 + 2i$, find the eigenvector:

   $$(3 - (1 + 2i))v_1 - 2v_2 = 0 \quad \Rightarrow \quad (2 - 2i)v_1 - 2v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$$

   Now, we compute $\mathrm{e}^{\lambda t}\mathbf{v}$. The solution is

   $$\mathbf{x}(t) = C_1 \mathrm{real}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2 \mathrm{imag}(\mathrm{e}^{\lambda t}\mathbf{v})$$

   which is shown in the back of the book.

   The solution to Exercise 3 is also given in the back of the text.

3. Exercises 1, 3, 7, pg. 429.

   NOTE: These solutions will differ from the format in the book, but they are equivalent. Here, we give the solutions as we did them in class.

   - Exercise 1: $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

     SOLUTION: We see that the trace is 2 and the determinant is 1, so

     $$\lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = 1, 1$$

     We don't need an eigenvector for the solution in this case, rather we compute $\mathbf{w}$:

     $$\begin{aligned} (3-1)x_0 - 4y_0 &= w_1 \\ x_0 + (-1-1)y_0 &= w_2 \end{aligned}$$

     Now the solution is given by

     $$\mathbf{x}(t) = \mathrm{e}^{t}\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 2x_0 - 4y_0 \\ x_0 - 2y_0 \end{bmatrix} \right)$$

- Exercise 3: $A = \begin{bmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{bmatrix}$

  SOLUTION: We see that the trace is -2 and the determinant is 1, so

  $$\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -1, -1$$

  We don't need an eigenvector for the solution in this case, rather we compute $\mathbf{w}$:

  $$\begin{aligned} ((-3/2) + 1)x_0 + y_0 &= w_1 \\ (-1/4)x_0 + ((-1/2) + 1)y_0 &= w_2 \end{aligned}$$

  Now the solution is given by

  $$\mathbf{x}(t) = \mathrm{e}^t \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} (-1/2)x_0 + y_0 \\ (-1/4)x_0 + (1/2)y_0 \end{bmatrix} \right)$$

- Exercise 7: $A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$ with $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

  SOLUTION: We see that the trace is -6 and the determinant is 9, so

  $$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad \lambda = -3, -3$$

  We don't need an eigenvector for the solution in this case, rather we compute $\mathbf{w}$:

  $$\begin{aligned} (1 - -3)(3) - 4(2) &= w_1 \\ 4(3) + (-7 - -3)(2) &= w_2 \end{aligned} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

  Now the solution is given by

  $$\mathbf{x}(t) = \mathrm{e}^{-3t} \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right)$$

4. Exercises 13, 15, pg 410 (Section 7.6, try to analyze with parameter- We'll do these more in depth later as well).

- Exercise 13: $A = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$, with $\begin{array}{c} \mathrm{Tr}(A) = 2\alpha \\ \det(A) = \alpha^2 + 1 \\ \Delta = 4\alpha^2 - 4(\alpha^2 + 1) = -4 \end{array}$

  SOLUTION: We see that the determinant is always positive, and the discriminant is always negative, which puts us "inside" the parabola in the Poincare classification.

  Therefore, we see that, if $\alpha > 0$, then we have a SPIRAL SOURCE. If $\alpha = 0$, we have a CENTER. If $\alpha < 0$, we have a SPIRAL SINK.

- Exercise 15: $A = \begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$, with $\begin{array}{c} \mathrm{Tr}(A) = 0 \\ \det(A) = -4 + 5\alpha \\ \Delta = -4(-4+5\alpha) = 16 - 20\alpha \end{array}$

  SOLUTION: If the trace is zero, that puts us on the axis labeled as the determinant (the vertical axis). Then, if $\alpha < 4/5$, the determinant is negative (SADDLE), if $\alpha = 4/5$, "uniform motion", and if $\alpha > 4/5$, then the determinant is positive and we have a "center".

5. Given the eigenvalues and eigenvectors for some matrix A, write the general solution to $\mathbf{x}' = A\mathbf{x}$. Furthermore, classify the origin as a sink, source, spiral sink, spiral source, saddle, or none of the above.

   (a) $\lambda = -1 + 2i \qquad \mathbf{v} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

   SOLUTION:

$$e^{\lambda t}\mathbf{v} = e^{-t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1-i \\ 2 \end{bmatrix} =$$

$$e^{-t} \begin{bmatrix} (\cos(2t) + \sin(2t)) + i(-\cos(2t) + \sin(2t)) \\ 2\cos(2t) + 2i\sin(2t) \end{bmatrix}$$

   Therefore, (factoring out the exponential),

$$\mathbf{x}(t) = e^{-t} C_1 \begin{bmatrix} cos(2t) + \sin(2t) \\ 2\cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} -\cos(2t) + \sin(2t) \\ 2\sin(2t) \end{bmatrix}$$

   (b) $\lambda = -2, 3 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   SOLUTION:

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   (c) $\lambda = -2, -2 \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

   SOLUTION: For us, we need the matrix to compute the solution (so we can write the solution in terms of the initial condition). This was left over from another section- Sorry!

   (d) $\lambda = 2, -3 \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   SOLUTION:

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

   (e) $\lambda = 1 + 3i \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

SOLUTION:

$$\mathbf{x}(t) = \mathrm{e}^t \left[C_1 \begin{bmatrix} \cos(3t) \\ \cos(3t) + \sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(3t) \\ -\cos(3t) + \sin(3t) \end{bmatrix} \right]$$

(f) $\lambda = 2i \qquad \mathbf{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

SOLUTION:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(2t) + \sin(2t) \\ \sin(2t) \end{bmatrix}$$