Quiz 6 (Take Home) Solutions

1. Suppose we have a harmonic oscillator with mass 1, spring constant 3 and damping coefficient 4 (and no forcing).

   (a) Solve the second order differential equation, if the initial position is 3 and initial velocity is -5.
   SOLUTION: The characteristic equation is $r^2 + 4r + 3 = 0$, so $r = -1, -3$.

   $$u(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-3t}$$

   Using the initial conditions, we solve the system:

   $$\begin{aligned} C_1 + C_2 &= 3 \\ -C_1 - 3C_2 &= -5 \end{aligned} \quad \Rightarrow \quad u(t) = 2\mathrm{e}^{-t} + \mathrm{e}^{-3t}$$

   (b) The oscillator is OVERDAMPED.

2. For each of the following, give your (final) guess for the form of the particular solution (do NOT solve for the coefficients).

   NOTE: The characteristic equation is same for all below, $r = -2, -4$.

   (a) $y'' + 6y' + 8y = t\sin(3t) + \cos(3t)$
   SOLUTION: Think of a polynomial of degree one times the sine/cosine:

   $$y_p = (At + B)\cos(3t) + (Ct + D)\sin(3t)$$

   (b) $y'' + 6y' + 8y = t^2 \mathrm{e}^{3t}$
   SOLUTION: $(At^2 + Bt + C)\mathrm{e}^{3t}$

   (c) $y'' + 6y' + 8y = \mathrm{e}^{-2t}$
   SOLUTION: We would use $y_p = A\mathrm{e}^{-2t}$, but that is part of the homogeneous solution, so $y_p = At\mathrm{e}^{-2t}$.

3. Solve: $y'' + 3y' + 2y = \cos(2t)$.

   Use the complex exponential to find the particular part of the solution.

   SOLUTION: We rewrite the equation as $y'' + 3y' + 2y = \mathrm{e}^{2it}$. The ansatz is then $y_p = A\mathrm{e}^{2it}$, which we subsitute into the DE:

   $$A\mathrm{e}^{2it}(-4 + 6i + 2) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-2+6i} = \frac{-2}{40} - \frac{6}{40}i = -\frac{1}{20} - \frac{3}{20}i$$

   Since we started with $\cos(2t)$, we want the real part of $A\mathrm{e}^{2it}$:

   $$\left(-\frac{1}{20} - \frac{3}{20}i\right)(\cos(2t) + i\sin(2t)) = \left(-\frac{1}{20}\cos(2t) + \frac{3}{20}\sin(2t)\right) + i\left(-\frac{3}{20}\cos(2t) - \frac{1}{20}\sin(2t)\right)$$

   Notice that you didn't need to compute the imaginary part; we included it for completeness. The particular solution is just the real part:

   $$y_p(t) = -\frac{1}{20}\cos(2t) + \frac{3}{20}\sin(2t)$$