Sample Questions (Chapter 3, Math 244)

1. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$

   (b) The characteristic equation for $y'' + xy' + \mathrm{e}^x y = 0$ is $r^2 + xr + \mathrm{e}^x = 0$

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.

   (d) Consider the function:
   $$y(t) = \cos(t) - \sin(t)$$
   Then amplitude is 1, the period is 1 and the phase shift is 0.

2. Find values of a for which **any** solution to:
   $$y'' + 10y' + ay = 0$$
   will tend to zero (that is, $\lim_{t\to 0} y(t) = 0$.

3. - Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
   - Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)

4. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

5. The following two parts go together- We're looking at periodic forcing of the undamped mass-spring system:

   (a) Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

   (b) Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:
   - Start over, with Method of Undetermined Coefficients
   - Take the limit of $u(t)$ from Question 5a as $\omega \to \omega_0$.

6. Given that $y_1 = \frac{1}{t}$ solves the differential equation:
   $$t^2 y'' - 2y = 0$$
   Find a fundamental set of solutions using Abel's Theorem.

7. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

8. Give the full solution. If there is an initial condition, solve the initial value problem.

   (a) $u'' + u = 3t + 4$, $u(0) = 0, u'(0) = 0$

   (b) $y'' + 2y' + 10y = 10t + 12 + 9\mathrm{e}^{-t}$

   (c) $y'' - 2y' + y = 4$, $y(0) = 1$, $y'(0) = 1$.

   (d) $y'' - 2y' + y = 3\cos(2t)$, $y(0) = 2$, $y'(0) = -1$.

   (e) $4y'' - 4y' + y = 16\mathrm{e}^{t/2}$

   (f) $y'' + 9y = \displaystyle\sum_{m=1}^{N} b_m \cos(m\pi t)$

9. For each differential equation below, write the final form of your ansatz for $y_p(t)$ using the Method of Undetermined Coefficients. Do **not** solve for the coefficients.

   (a) $y'' + 4y = 3\sin(2t)$

   (b) $y'' + 4y' + 5y = t^2 e^{3t} + 6te^{-2t}\sin(t)$

10. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

11. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

12. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2 e^{-t} + \frac{1}{2}t^2 - t$$

13. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

14. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.
    If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

15. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

16. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

17. A mass of 0.5 kg stretches a spring an additional 0.05 meters to get to equilibrium. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

18. A mass of $\frac{1}{2}$ kg is attached to a spring with spring constant 2 (kg/sec^2). The spring is pulled down an additional 1 meter then released. Find the equation of motion if the damping constant is $c = 2$ as well:

19. Given that $y_1(t) = (t+2)e^t$ and $y_2(t) = e^t - 2$ are both solutions of a certain DE of the form:

$$y'' + p(t)y' + q(t)y = 0$$

    then answer each question below, with a short reason.

   (a) Show that $W(y_1, y_2)(t) \neq 0$.

   (b) True or False: y_1, y_2 form a fundamental set of solutions.

   (c) True or False: $y_3(t) = (t+3)e^t - 2$ is also a solution.

   (d) True or False: $y_4(t) = te^t + 4$ is also a solution.

   (e) True or False: $y_5(t) = (t+1)e^t$ is also a solution.

   (f) Suppose that $y_p(t) = -5\cos(2t)$ is a solution to $y'' + p(t)y' + q(t)y = g(t)$. Give the general solution:

20. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE). HINT: Think about what kind of behavior you should expect- Is it a purely periodic function? Resonance? etc.

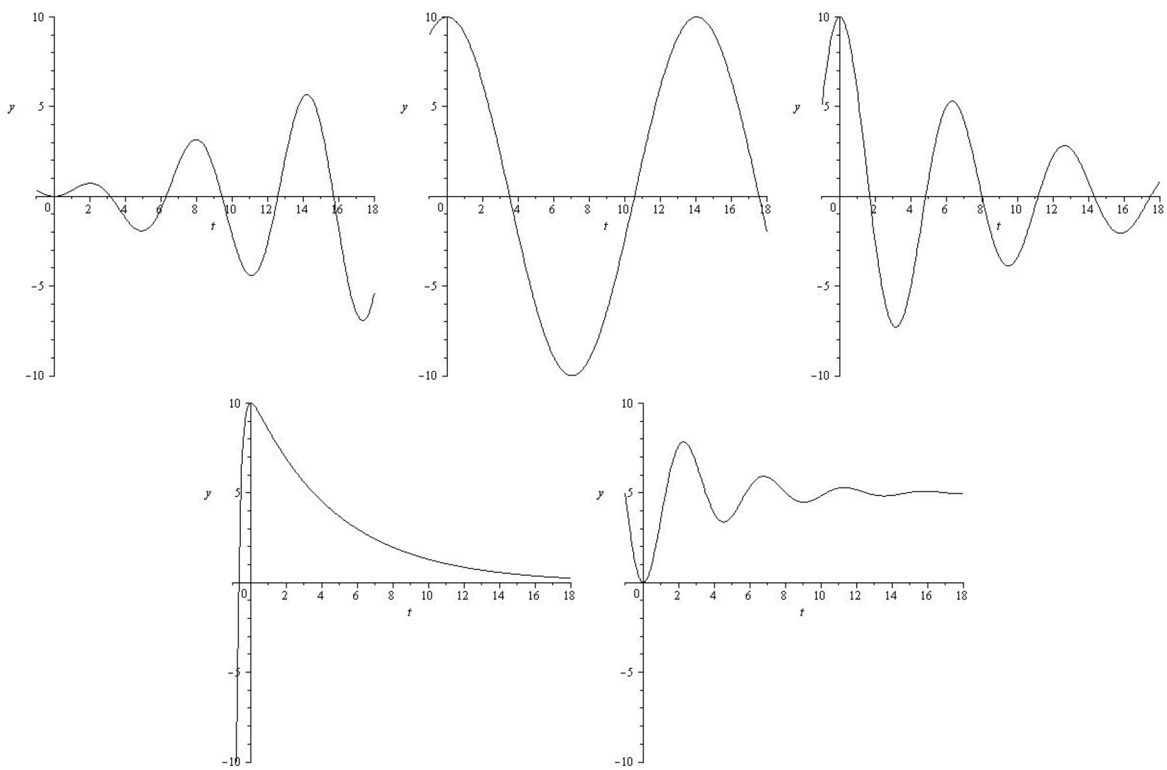
(a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$

(b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$

(c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$

(d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$

(e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$



21. Be sure that you understand all of the homework problems from the Section 3.8 handout.