Sample Question Solutions (Chapter 3, Math 244)

1. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$
   SOLUTION: False. The characteristic equation is for the homogeneous equation, $r^2 + r + 1 = 0$

   (b) The characteristic equation for $y'' + xy' + \mathrm{e}^x y = 0$ is $r^2 + xr + \mathrm{e}^x = 0$
   SOLUTION: False. The characteristic equation was defined only for DEs with constant coefficients, since our ansatz depended on constant coefficients.

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.
   SOLUTION: True. It is true generally- If L is a linear operator, then $L(0) = 0$.

   (d) Consider the function:
   $$y(t) = \cos(t) - \sin(t)$$
   Then amplitude is 1, the period is 1 and the phase shift is 0.
   SOLUTION: False. For this question to make sense, we have to first write the function as $R\cos(\omega(t - \delta))$. In this case, the amplitude is R:
   $$R = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
   The period is 2π (the circular frequency, or natural frequency, is 1), and the phase shift δ is:
   $$\tan(\delta) = -1 \quad \Rightarrow \quad \delta = -\frac{\pi}{4}$$

2. Find values of a for which **any** solution to:
   $$y'' + 10y' + ay = 0$$
   will tend to zero (that is, $\lim_{t\to\infty} y(t) = 0$.
   SOLUTION: Use the characteristic equation and check the 3 cases (for the discriminant). That is,
   $$r^2 + 10r + a = 0 \quad \Rightarrow \quad r = \frac{-10 \pm \sqrt{100 - 4a}}{2}$$
   We check some special cases:

   - If $100 - 4a = 0$ (or $a = 25$), we get a double root, $r = -5, -5$, or $y_h = \mathrm{e}^{-5t}(C_1 + C_2 t)$, and all solutions tend to zero.
   - If $100 - 4a < 0$, or $a > 25$, then the roots are complex, and we can write $r = -5 \pm \beta i$. The solution is then
   $$y_h = \mathrm{e}^{-5t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$
   and again, this will tend to zero for any choice of C_1, C_2.
   - In the case that $a < 25$, we have to be a bit careful. While it is true that both roots will be *real*, we also want them to both be *negative* for all solutions to tend to zero.
     - When will they both be negative? If $100 - 4a < 100$ (or $\sqrt{100 - 4a} < 10$). This happens as long as $a > 0$.
     - If $a = 0$, the roots will be $r = -10, 0$, and $y_h = C_1 \mathrm{e}^{-10t} + C_2$- Therefore, I could choose $C_1 = 0$ and $C_2 \neq 0$, and my solution will not go to zero.
     - If $a < 0$, the roots will be mixed in sign (one positive, one negative), so the solutions will not all tend to zero.

CONCLUSION: If $a > 0$, all solutions to the homogeneous will tend to zero.

Side Remark: This analysis was similar to what we were discussing in class- given a mass spring model, $mu'' + \gamma u' + ku = 0$ where the coefficients were all greater than zero, then all solutions go to zero.

3.
- Compute the Wronskian between $f(x) = \cos(x)$ and $g(x) = 1$.
  SOLUTION: $W(\cos(x), 1) = \sin(x)$
- Can these be two solutions to a second order linear homogeneous differential equation? Be specific. (Hint: Abel's Theorem)
  SOLUTION: Abel's Theorem tells us that the Wronskian between two solutions to a second order linear homogeneous DE will either be identically zero or never zero on the interval on which the solution(s) are defined.

  Therefore, as long as the interval for the solutions do not contain a multiple of π (for example, $(0, \pi)$, $(\pi, 2\pi)$, etc), then it is possible for the Wronskian for two solutions to be $\sin(x)$.

4. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

   SOLUTION: The operator is found by getting all terms in y to one side of the equation, everything else on the other. In this case, we have:

$$L(y) = y' - y^2$$

   This is not a linear operator. We can check using the definition:

$$L(cy) = cy' - c^2y^2 \neq cL(y)$$

   Furthermore,

$$L(y_1 + y_2) = (y_1' + y_2') - (y_1 + y_2)^2 \neq L(y_1) + L(y_2)$$

5. These two problems go together, and show what's happening as beating becomes resonance.

   (a) Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t)$, $\quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.
   SOLUTION: The characteristic equation is: $r^2 + \omega_0^2 = 0$, or $r = \pm\omega_0 i$. Therefore,

$$u_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

   Using the Method of Undetermined Coefficients, $u_p = A\mathrm{e}^{i\omega t}$. Substitution into the DE:

$$-\omega^2 A\mathrm{e}^{i\omega t} + \omega_0^2 A\mathrm{e}^{i\omega t} = F_0\mathrm{e}^{i\omega t} \quad \Rightarrow \quad A = \frac{F_0}{\omega_0^2 - \omega^2}$$

   This expression is real, so the particular solution is this constant times the cosine. Putting it all together so far, the general solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2} \cos(\omega t)$$

   Put in the initial conditions $u(0) = 0$ and $u'(0) = 0$ to see that $C_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$ and $C_2 = 0$ so that

$$u(t) = \frac{F_0(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2}$$

   (b) Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:

- Start over, with Method of Undetermined Coefficients
  SOLUTION: Let $u_p = Ate^{i\omega_0 t}$. Then

  $$u_p' = Ae^{i\omega_0 t} + i\omega_0 Ate^{i\omega_0 t} \qquad u_p'' = 2i\omega_0 Ae^{i\omega_0 t} - \omega_0^2 Ate^{i\omega_0 t}$$

  Now,

  $$u_p'' + \omega_0^2 u_p = 2i\omega Ae^{i\omega t} = F_0 e^{i\omega_0 t} \quad \Rightarrow \quad A = \frac{F_0}{2i\omega_0} = -\frac{F_0}{2\omega_0}\, i$$

  We want the real part of $Ate^{i\omega_0 t}$:

  $$Ate^{i\omega_0\, t} = -\frac{F_0}{2\omega_0}\, it(\cos(\omega_0\, t) + i\sin(\omega_0 t))$$

  which we see is:

  $$u_p = \frac{F_0}{2\omega_0} t\, \sin(\omega_0\, t)$$

  Putting it all together,

  $$u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{2\omega_0} t\sin(\omega_0 t)$$

  Now, $u(0) = 0$ means that $C_1 = 0$. Differentiating for the second IC,

  $$u' = \omega_0 C_2\cos(\omega_0\, t) + \frac{F_0}{2\omega_0}\sin(\omega_0 t) + \frac{F_0\omega_0}{2\omega_0} t\cos(\omega_0 t)$$

  In this case, $C_2 = 0$ as well, so the particular solution is the full solution.
- Take the limit of $u(t)$ from Question 5a as $\omega \to \omega_0$.
  SOLUTION:

  $$\lim_{\omega\to\omega_0} \frac{F_0(\cos(\omega t) - \cos(\omega_0 t)}{\omega_0^2 - \omega^2} =?$$

  We can use l'Hospital's Rule (differentiate with respect to ω!):

  $$= \lim_{\omega\to\omega_0} \frac{-F_0 t\sin(\omega t)}{-2\omega} = \frac{F_0}{2\omega_0}\, t\, \sin(\omega_0 t)$$

6. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

   $$t^2 y'' - 2y = 0$$

   Find a fundamental set of solutions using Abel's Theorem:

   SOLUTION: First, rewrite the differential equation in standard form:

   $$y'' - \frac{2}{t^2} y = 0$$

   Then $p(t) = 0$ and $W(y_1, y_2) = Ce^0 = C$. On the other hand, the Wronskian is:

   $$W(y_1, y_2) = \frac{1}{t} y_2' + \frac{1}{t^2} y_2$$

   Put these together:

   $$\frac{1}{t} y_2' + \frac{1}{t^2} y_2 = C \quad y_2' + \frac{1}{t} y_2 = Ct$$

   The integrating factor is t,

   $$(ty_2)' = Ct^2 \quad \Rightarrow \quad ty_2 = C_1 t^3 + C_2 \quad \Rightarrow \quad C_1 t^2 + \frac{C_2}{t}$$

   Notice that we have *both* parts of the homogeneous solution, $y_1 = \frac{1}{t}$ and $y_2 = t^2$.

7. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

   SOLUTION: We can find the differential equation:

$$0.01u'' + 0.05u' + ku = 0 \quad \Rightarrow \quad u'' + 5u' + 100ku = 0$$

   Then the system is *critically damped* if the discriminant (from the quadratic formula) is zero:

$$b^2 - 4ac = 25 - 4 \cdot 100k = 0 \quad \Rightarrow \quad k = \frac{25}{400} = \frac{1}{16}$$

   The system is *underdamped* if the discriminant is negative:

$$25 - 400k < 0 \quad \Rightarrow \quad k > \frac{1}{16}$$

8. Give the full solution.

   (a) $u'' + u = 3t + 4$, $u(0) = 0$, $u'(0) = 0$.
   SOLUTION: For the homogeneous part, $r^2 + 1 = 0$, so $r = \pm i$ and $u_h(t) = C_1 \cos(t) + C_2 \sin(t)$. For the particular part, by the Method of Undetermined Coefficients,

$$u_p = At + B, \quad u_p = A, \quad u_p'' = 0 \quad \Rightarrow \quad 0 + (At + B) = 3t + 4$$

   Therefore,

$$u(t) = C_1 \cos(t) + C_2 \sin(t) + 3t + 4$$

   Solve for C_1, C_2:

$$\begin{array}{ll} 0 & = C_1 + 0 + 0 + 4 \\ 0 & = 0 + C_2 + 3 \end{array} \quad \Rightarrow \quad u(t) = -4\cos(t) - 3\sin(t) + 3t + 4$$

   (b) $y'' + 2y' + 10y = 10t + 12 + 9\mathrm{e}^{-t}$
   SOLUTION: We break the problem up. First, we'll solve $y'' + 2y' + 10y = 0$, then we'll solve $y'' + 2y' + 10y = 10t + 12$, and finally solve $y'' + 2y' + 10y = 9\mathrm{e}^{-9t}$.

   - $y'' + 2y' + 10y = 0$
     Complete the square to solve:

$$r^2 + 2r + 10 = r^2 + 2r + 1 + 9 = 0 \quad \Rightarrow \quad (r+1)^2 = -9 \quad \Rightarrow \quad r = -1 \pm 3i$$

     so that

$$y_h(t) = \mathrm{e}^{-t}(C_1 \cos(3t) + C_2 \sin(3t))$$

   - $y'' + 2y' + 10y = 10t + 12$.
     Guess that $y_{p_1}(t) = At + B$. Substituting, we get

$$0 + 2A + 10(At + B) = 10t + 12$$

     and now equate the coefficients:

$$\begin{array}{lrl} t: & 10A & = 10 \\ \text{const} & 2A + 10B & = 12 \end{array} \quad \Rightarrow \quad A = 1, B = 1 \quad \Rightarrow \quad y_{p_1}(t) = t + 1$$

   - Finally, guess that $y_{p_2} = A\mathrm{e}^{-t}$ and substitute back into the DE to get

$$A\mathrm{e}^{-t}(1 - 2 + 10) = 9\mathrm{e}^{-t} \quad \Rightarrow \quad y_{p_2}(t) = \mathrm{e}^{-t}$$

The full solution is given by

$$y(t) = e^{-t}(C_1 \cos(3t) + C_2 \sin(3t)) + t + 1 + e^{-t}$$

(c) $y'' - 2y' + y = 4$, $y(0) = 1$, $y'(0) = 1$
SOLUTION: $r^2 - 2r + 1 = (r-1)^2$, or $r = 1, 1$. The homogeneous part is $y_h(t) = e^t(C_1 + C_2 t)$. We'll guess $y_p = A$, and by inspection we see that $y_p = 4$, so the solution is

$$y(t) = e^t(C_1 + C_2 t) + 4$$

Solving for the constants will require the derivative: $y' = e^t(C_1 + C_2 t) + e^t C_2$, and:

$$\begin{aligned} 1 &= C_1 + 4 \\ 1 &= C_1 + C_2 \end{aligned} \quad \Rightarrow \quad C_1 = -3, C_2 = 4 \quad \Rightarrow \quad y(t) = e^t(-3 + 4t) + 4$$

(d) $y'' - 2y' + y = 3\cos(2t)$, $y(0) = 2$, $y'(0) = -1$
SOLUTION: The homogeneous part of the solution is the same as the previous question. We'll complexify the problem to solve it, so that $y_p = Ae^{2it}$:

$$Ae^{2it}(-4 - 2(2i) + 1) = 3e^{2it} \quad \Rightarrow \quad A = \frac{3}{-3 - 4i}$$

The solution is given by the real part of Ae^{2it}:

$$\mathrm{Re}\left(\frac{3(-3+4i)}{9+16}(\cos(2t) + i\sin(2t))\right) = -\frac{9}{25}\cos(2t) - \frac{12}{25}\sin(2t)$$

For the full solution,

$$y(t) = e^{-t}(C_1 + C_2 t) - \frac{9}{25}\cos(2t) - \frac{12}{25}\sin(2t)$$

And we should find that

$$y(t) = e^{-t}\left(\frac{59}{25} - \frac{12}{5}t\right) - \frac{9}{25}\cos(2t) - \frac{12}{25}\sin(2t)$$

(Sorry about the fractions!)

(e) $4y'' - 4y' + y = 16e^{t/2}$
SOLUTION: For the homogeneous part of the solution,

$$4r^2 - 4r + 1 = 0 \quad \Rightarrow \quad r^2 - r + \frac{1}{4} = 0 \quad \Rightarrow \quad \left(r - \frac{1}{2}\right)^2 = 0 \quad \Rightarrow \quad r = 1/2, 1/2$$

Therefore, the homogeneous part of the solution is $e^{t/2}(C_1 + C_2 t)$. Looking at the particular solution, we'll need to multiply by t^2 since $e^{t/2}$ and $te^{t/2}$ are both already in the homogeneous part of the solution. Therefore, y_p and the simplified derivatives are:

$$y_p(t) = At^2 e^{t/2} \quad y_p' = Ae^{t/2}\left(\frac{1}{2}t^2 + 2t\right) \qquad y_p'' = Ae^{t/2}\left(\frac{1}{4}t^2 + 2t + 2\right)$$

Substituting these back into the DE, we get

$$Ae^{t/2}\left(4\left(\frac{1}{4}t^2 + 2t + 2\right) - 4\left(\frac{1}{2}t^2 + 2t\right) + t^2\right) = 16e^{t/2}$$

Simplifying, we see $A = 2$ and the full general solution is given by

$$y(t) = e^{t/2}(C_1 + C_2 t + 2t^2)$$

(f) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

The homogeneous part of the solution is $C_1\cos(3t) + C_2\sin(3t)$. We see that $3 \neq m\pi$ for $m = 1, 2, 3, \ldots$.

The forcing function is a sum of N functions, the m^{th} function is:

$$g_m(t) = b_m\cos(m\pi t) \quad \Rightarrow \quad y_{p_m} = A\mathrm{e}^{im\pi t}$$

Differentiating,

$$y_{p_m}'' = -m^2\pi^2 A\mathrm{e}^{im\pi t} \quad \Rightarrow \quad y_p'' + 9y_p = (9 - m^2\pi^2)A\mathrm{e}^{im\pi t} = b_m\mathrm{e}^{im\pi t}$$

Therefore, $A = b_m/(9 - m^2\pi^2)$, and the full solution is:

$$y(t) = C_1\cos(3t) + C_2\sin(3t) + \sum_{m=1}^{N} \frac{b_m}{9 - m^2\pi^2}\cos(m\pi t)$$

9. For each differential equation below, write the final form of your ansatz for $y_p(t)$ using the Method of Undetermined Coefficients. Do **not** solve for the coefficients.

(a) $y'' + 4y = 3\sin(2t)$

The homogeneous part of the solution is $C_1\cos(2t) + C_2\sin(2t)$, so we'll need to multiply by t. We can write the guess one of two ways (either will be OK):

- $y_p = (A\cos(2t) + B\sin(2t)) \cdot t$
- For the complexified problem, $y_p = At\mathrm{e}^{2it}$, where A may be complex.

(b) $y'' + 4y' + 5y = t^2\mathrm{e}^{3t} + 6t\mathrm{e}^{-2t}\sin(t)$

The homogeneous part is $\mathrm{e}^{-2t}(C_1\cos(t) + C_2\sin(t))$. For our first guess, we can take $y_{p_1}(t) = (At^2 + Bt + C)\mathrm{e}^{3t}$.

For the second guess, we can write it one of two ways (either is OK):

- $y_{p_2}(t) = ((At + B)\mathrm{e}^{-2t}\cos(t) + (Ct + D)\mathrm{e}^{-2t}\sin(t)) \cdot t$
- $y_{p_2}(t) = (At + B)\mathrm{e}^{(-2+i)t} \cdot t$, where A, B could be complex.

10. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $\mathrm{e}^{(3-2i)t}$ (iii) $\mathrm{e}^{i\pi}$

NOTE for the SOLUTION: Remember that for any non-negative number A, we can write $A = \mathrm{e}^{\ln(A)}$.

- $2^{i-1} = \mathrm{e}^{ln(2^{i-1})} = \mathrm{e}^{(i-1)\ln(2)} = \mathrm{e}^{-\ln(2)}\mathrm{e}^{i\ln(2)} = \frac{1}{2}\left(\cos(\ln(2)) + i\sin(\ln(2))\right)$
- $\mathrm{e}^{(3-2i)t} = \mathrm{e}^{3t}\mathrm{e}^{-2ti} = \mathrm{e}^{3t}\left(\cos(-2t) + i\sin(-2t)\right) = \mathrm{e}^{3t}\left(\cos(2t) - i\sin(2t)\right)$

  (Recall that cosine is an even function, sine is an odd function).
- $\mathrm{e}^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

11. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

SOLUTIONS:

(i) $r = \sqrt{1+3} = 2$, $\theta = -2\pi/3$ (look at its graph, use 30-60-90 triangle):

$$-1 - \sqrt{3}i = 2\mathrm{e}^{-\frac{2\pi}{3}i}$$

(ii) $3i = 3\mathrm{e}^{\frac{\pi}{2}i}$

(iii) $-4 = 4\mathrm{e}^{\pi i}$

(iv) $\sqrt{3} - i = 2\mathrm{e}^{-\frac{\pi}{6}i}$

12. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2\mathrm{e}^{-t} + \frac{1}{2}t^2 - t$$

SOLUTION: Work backwards from the characteristic equation to build the homogeneous DE (then figure out the rest):

The roots to the characteristic equation are $r = 0$ and $r = -1$. The characteristic equation must be $r(r+1) = 0$ (or a constant multiple of that). Therefore, the differential equation is:

$$y'' + y' = 0$$

For $y_p = \frac{1}{2}t^2 - t$ to be the particular solution,

$$y_p'' + y_p' = (1) + (t-1) = t$$

so the full differential equation must be:

$$y'' + y' = t$$

13. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

SOLUTION: Write in standard form first-

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

The coefficient functions are all continuous on each of three intervals:

$$(-\infty, 0), (0, 4) \text{ and } (4, \infty)$$

Since the initial time is 3, we choose the middle interval, $(0, 4)$.

14. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

If $L(3\mathrm{e}^{2t}) = -9\mathrm{e}^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + \mathrm{e}^{2t}$$

SOLUTION: This purpose of this question is to see if we can use the properties of linearity to get at the answer.

We see that: $L(3\mathrm{e}^{2t}) = -9\mathrm{e}^{2t}$, or $L(\mathrm{e}^{2t}) = -3\mathrm{e}^{2t}$ so:

$$L\left(-\frac{1}{3}\mathrm{e}^{2t}\right) = \mathrm{e}^{2t}$$

And for the second part,

$$L(t^2 + 3t) = 5t^2 + 3t - 16 \quad \Rightarrow \quad L((-2)(t^2 + 3t)) = -10t^2 + 6t - 32$$

The particular solution is therefore:

$$y_p(t) = -2(t^2 + 3t) - \frac{1}{3}\mathrm{e}^{2t}$$

since we have shown that

$$L\left(-2(t^2 + 3t) - \frac{1}{3}\mathrm{e}^{2t}\right) = -10t^2 + 6t - 32 + \mathrm{e}^{2t}$$

15. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$$

Using Abel's Theorem (and writing the DE in standard form first):

$$y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y = 0$$

Therefore,

$$W(y_1, y_2) = Ce^{-\int \frac{1}{x}\,dx} = \frac{C}{x}$$

16. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

SOLUTION: Solving as usual gives:

$$y(t) = \left(\frac{3-\alpha}{5}\right)e^{-2t} + \left(\frac{\alpha+2}{5}\right)e^{3t}$$

so to make sure the solutions tend to zero, $\alpha = -2$ (to zero out the second term).

17. A mass of 0.5 kg stretches a spring an additional 0.05 meters to get to equilibrium. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

SOLUTION:

We use Hooke's Law at equilibrium: $mg - kL = 0$, or

$$k = \frac{mg}{L} = \frac{4.9}{0.05} = 98$$

For the second part, a stiff spring will not stretch, so L will be small (and k would therefore be large), and a spring that is not stiff will stretch a great deal (so that k will be smaller).

18. A mass of $\frac{1}{2}$ kg is attached to a spring with spring constant 2 (kg/sec^2). The spring is pulled down an additional 1 meter then released. Find the equation of motion if the damping constant is $c = 2$ as well:

SOLUTION: Just substitute in the values

$$\frac{1}{2}u'' + 2u' + 2u = 0$$

Pulling down the spring and releasing: $u(0) = 1$, $u'(0) = 0$ (Down is positive)

19. Given that $y_1(t) = (t+2)e^t$ and $y_2(t) = e^t - 2$ are both solutions of a certain DE of the form:

$$y'' + p(t)y' + q(t)y = 0$$

then answer each question below, with a short reason.

(a) Show that $W(y_1, y_2)(t) \neq 0$.

SOLUTION: You don't need to expand out the whole thing- Look at your expression and see that it does not simplify to zero.

(b) True or False: y_1, y_2 form a fundamental set of solutions.

True by part (a).

Side Remark: The next few questions are using the Superposition Principle- That any linear combination of the homogeneous solutions will give you another homogeneous solution...

(c) True or False: $y_3(t) = (t+3)\mathrm{e}^t - 2$ is also a solution.

SOLUTION: Let's see if a linear combination of the two solutions we are given work out to y_3:

$$c_1(t+2)\mathrm{e}^t + c_2(\mathrm{e}^t - 2) = \mathrm{e}^t(c_1 t + 2c_1 + c_2) - 2c_2 = (t+3)\mathrm{e}^t - 2$$

If this were true, then $c_2 = 1$, and $c_1 = 1$- Yes, it is true.

(d) True or False: $y_4(t) = t\mathrm{e}^t + 4$ is also a solution.

SOLUTION: Using C_1, C_2 from the previous part, this would mean that $C_2 = -2$, but then $C_1 = 2$, and that would leave $2t\mathrm{e}^t$ rather than $t\mathrm{e}^{2t}$, so false.

(e) True or False: $y_5(t) = (t+1)\mathrm{e}^t$ is also a solution.

SOLUTION: Do you get the idea now? In this case, $C_2 = 0$, then $C_1 = \frac{1}{2}$, but that wouldn't get us what we want, either. False.

(f) Suppose that $y_p(t) = -5\cos(2t)$ is a solution to $y'' + p(t)y' + q(t)y = g(t)$. Give the general solution:

$$C_1(t+2)\mathrm{e}^t + C_2(\mathrm{e}^t - 2) - 5\cos(2t)$$

20. Match the solution curve to its IVP (There is one DE with no graph, and one graph with no DE- You should not try to completely solve each DE).

(a) $5y'' + y' + 5y = 0$, $y(0) = 10$, $y'(0) = 0$ (Complex roots, solutions go to zero) Graph C

(b) $y'' + 5y' + y = 0$, $y(0) = 10$, $y'(0) = 0$ (Exponentials, solutions go to zero) Graph D

(c) $y'' + y' + \frac{5}{4}y = 0$, $y(0) = 10$, $y'(0) = 0$ NOT USED

(d) $5y'' + 5y = 4\cos(t)$, $y(0) = 0$, $y'(0) = 0$ (Pure Harmonic) Graph B

(e) $y'' + \frac{1}{2}y' + 2y = 10$, $y(0) = 0$, $y'(0) = 0$ (Complex roots to homogeneous solution, constant particular solution) Graph E

SOLUTION: If the graphs are labeled: Top row: A, B, second row: C, D, and last row E, then the graphs are given above.