# Addition to 3.5: The Complex Exponential

In the Method of Undetermined Coefficients, if the forcing function is an exponential function, $g(t) = \mathrm{e}^{at}$, then our guess is straightforward- $y_p(t) = A\mathrm{e}^{at}$, then we substitute this into the DE and solve for $A$. For example, given $y'' + 2y' + y = \mathrm{e}^{2t}$, the homogeneous part of the solution has $r = -1, -1$, so that

$$y_h(t) = C_1\mathrm{e}^{-t} + C_2 t\mathrm{e}^{-t} \text{ and } y_p = A\mathrm{e}^{2t}, \quad y_p' = 2A\mathrm{e}^{2t}, \quad y_p'' = 4A\mathrm{e}^{2t}$$

Substituting $y_p$ into the DE, we solve for $A$ (I've also factored out the common term):

$$A\mathrm{e}^{2t}(4 + 2(2) + 1) = \mathrm{e}^{2t} \quad \Rightarrow \quad A = \frac{1}{9}$$

Therefore, $y_p = \frac{1}{9}\mathrm{e}^{2t}$.

If the right-hand side is a sine and/or cosine, we can take an ansatz as a sum of the sine and cosine, but it turns out to be convenient to use the complex exponential instead. Here's an example:

**Example 1**

Solve $y'' + 2y' + y = \cos(3t)$.

SOLUTION: We already know how to solve the homogeneous problem ($y_h$ is given above). We want to focus now on the particular solution, $y_p(t)$- We will embed the given DE into a bigger problem:

$$y'' + 2y' + y = \cos(3t) + i\sin(3t) = \mathrm{e}^{3it}$$

We don't need the full solution- Because $\cos(3t)$ is the real part of the exponential, we will solve the DE and only use the real part as our answer. In doing this, we've changed the forcing function from "trig" type to "exponential type", so our ansatz is now $y_p = A\mathrm{e}^{3it}$.

Continuing, treat the forcing function like a regular exponential and solve for the unknown constant $A$:

$$y_p = A\mathrm{e}^{3it} \qquad y_p' = 3iA\mathrm{e}^{3it} \qquad y_p'' = -9A\mathrm{e}^{3it}.$$

Substituting into the DE, we get:

$$A\mathrm{e}^{3it}(-9 + 2(3i) + 1) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{-8+6i} = \frac{1}{2}\,\frac{1}{-4+3i}$$

where I've factored out the 1/2 to make the computations easier.

Our solution is the real part of $A\mathrm{e}^{3it}$, so we compute that to get our answer. Leaving off the 1/2,

$$\frac{1}{-4+3i}\left(\cos(3t) + i\sin(3t)\right) = \frac{-4-3i}{25}\left(\cos(3t) + i\sin(3t)\right)$$

If we multiply this out, we get:

$$\left(-\frac{4}{25}\cos(3t) + \frac{3}{25}\sin(3t)\right) + i\left(-\frac{3}{25}\cos(3t) - \frac{4}{25}\sin(3t)\right)$$

We now have our answer- Note that we did not actually need to compute the imaginary part, we did so in this case to show the complete computation, if you needed it. In our case, we only need the real part, so the particular solution is the following (remember to put the 1/2 back in)

$$y_p = \text{Real}(A\mathrm{e}^{3it}) = -\frac{4}{50}\cos(3t) + \frac{3}{50}\sin(3t)$$

**Example 2**

Solve: $y'' + 2y' + y = \sin(3t)$

In this case, we would use exactly the same technique as before, but we go after the imaginary part of the solution at the end. That is, we:

- Replace $\sin(3t)$ by $e^{3it}$
- Use $y_p = Ae^{3it}$.
- Take the imaginary part of $y_p$ as the solution.

And we can see what our solution is:

$$y_p = -\frac{3}{50}\cos(3t) - \frac{4}{50}\sin(3t)$$

**Example 3 (We have to multiply by $t$)**

Solve: $y'' + 9y = \sin(3t)$. In this case, $\sin(3t)$ is part of the homogeneous solution, so our guess needs to take that into account. We re-write the ODE as:

$$y'' + 9y = e^{3it}$$

And our ansatz will be:

$$y_p = Ate^{3it} \qquad y_p' = Ae^{3it} + 3iAte^{3it} \qquad y_p'' = 6iAe^{3it} - 9Ate^{3it}$$

Solving for our constant:

$$y_p'' + 9y_p = 6iAe^{3it} \quad \Rightarrow \quad 6iAe^{3it} = e^{3it}.$$

From this, we see that:

$$A = \frac{1}{6i} = -\frac{1}{6}i$$

As before, we want only the imaginary part of $Ae^{3it}$, which in this case will be:

$$y_p = \text{Imag}(Ate^{3it}) = \text{Imag}\left(-\frac{t}{6}\,i(\cos(3t) + i\sin(3t)\right) = \frac{t}{6}\sin(3t)$$

## Trigonometry and a Shortcut

In the exercises, we will show that, if

$$F(t) = \frac{1}{\alpha + i\beta}e^{i\omega t}$$

then the real part of $F(t)$ can be written as:

$$\text{Re}(F(t)) = R\cos(\omega t - \delta), \qquad \text{where } R = \frac{1}{|\alpha + i\beta|}, \quad \delta = \tan^{-1}(\beta/\alpha)$$

Similarly, the imaginary part of $F(t)$ can be written as:

$$\text{Im}(F(t)) = R\cos(\omega t - \delta), \qquad \text{where } R = \frac{1}{|\alpha + i\beta|}, \quad \delta = \tan^{-1}(-\alpha/\beta)$$

These formulae are very useful if we want to avoid a lot of tedious computation. Here are some examples.

**Example**

Find the particular solution, and express it as $R\cos(\omega t - \delta)$.

$$y'' + y' - 2y = \cos(2t)$$

SOLUTION: Take the right side as $\mathrm{e}^{i2t}$, and the ansatz (with derivatives) is:

$$y_p = A\mathrm{e}^{2it} \qquad y_p' = 2iA\mathrm{e}^{2it} \qquad y_p'' = -4A\mathrm{e}^{2it}$$

Putting these into the DE and solve for $A$:

$$A\mathrm{e}^{2it}\left(-4 + 2i - 2(1)\right) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-6+2i}$$

Now, recall our formulas- We have our complex number, so find its magnitude and argument:

$$|\alpha + i\beta| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10} \qquad \delta = \tan^{-1}(-1/3) + \pi$$

(We added $\pi$ since $(-6, 2)$ is in Quadrant II). The particular solution is then:

$$\frac{1}{2\sqrt{10}}\cos\left(2t - (\tan^{-1}(-1/3) + \pi)\right)$$

**Example**

Find the particular solution and express it as $R\cos(\omega t - \delta)$.

$$y'' - 3y' + 2y = \sin(3t)$$

SOLUTION: Write the ansatz as $y_p = A\mathrm{e}^{3it}$, and substitute it into the DE like last time:

$$A\mathrm{e}^{3it}(-9 - 3(3i) + 2) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{-7-9i}$$

From this, $|-7-9i| = \sqrt{49+81} = \sqrt{130}$ and $\delta = \tan^{-1}(-7/9)$ (We're using the imaginary part of the solution this time).

$$y_p = \frac{1}{\sqrt{130}}\cos(3t - \delta)$$

# Exercises:

1. For each of the following, write the particular solution as $y_p = R\cos(\omega t - \delta)$.
   (a) $y'' + 7y = 3\cos(3t)$
   (b) $y'' + y' + 3y = 2\sin(2t)$
   (c) $y'' + 2y' + y = \cos(2t)$
   (d) $y'' + 2y' + 2y = \cos(t)$
2. Use the complexification technique to find the particular solution to $y'' + y = \mathrm{e}^{-t}\cos(t)$.
   There will be a bit of algebra involved, but not too bad- Write $g(t)$ as $\mathrm{e}^{(\alpha+\beta i)t}$
3. If $F(t) = \frac{1}{\alpha+i\beta}\mathrm{e}^{i\omega t}$, then show (by direction computation) that the real part of $F$ can be expressed as

$$\mathrm{Re}(F(t)) = \frac{1}{|\alpha+i\beta|}\cos(\omega t - \delta), \qquad \text{where } \delta = \arg(\alpha + i\beta)$$

   And the imaginary part as:

$$\mathrm{Im}(F(t)) = \frac{1}{|\alpha+i\beta|}\cos(\omega t - \delta), \qquad \text{where } \delta = \arg(-\beta + i\alpha)$$