Solutions to the Complex Ansatz Exercises

1. Solve for the particular solution and write it as $R\cos(\omega t - \delta)$

   (a) $y'' + 7y = 3\cos(3t)$
   SOLUTION: First write the "larger" problem as $y'' + 7y = 3\mathrm{e}^{3it}$. The ansatz will be $y_p = A\mathrm{e}^{3it}$, but we'll only use the real part. Substitute into the DE and factor out $A\mathrm{e}^{3it}$:

   $$A\mathrm{e}^{3it}(-9+7) = 3\mathrm{e}^{3it} \quad \Rightarrow \quad A = -\frac{3}{2} \quad \Rightarrow \quad y_p(t) = \mathrm{Re}\left(-\frac{3}{2}\mathrm{e}^{3it}\right),$$

   so the particular part of the solution is $y_p = -\frac{3}{2}\cos(3t)$.

   (b) $y'' + y' + 3y = 2\sin(2t)$
   SOLUTION: First rewrite as $y'' + y' + 3y = 2\mathrm{e}^{2it}$, and the ansatz will be $y_p = A\mathrm{e}^{2it}$ (imaginary part). Substitute:

   $$A\mathrm{e}^{2it}(-4+2i+3)\mathrm{e}^{3it} = 2\mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{2}{-1+2i}$$

   We take the imaginary part of the expression below (the complex number has been rationalized):

   $$\frac{-2-4i}{5}(\cos(2t) + i\sin(2t)) \quad \Rightarrow \quad y_p = \frac{2}{5}\left(-2\cos(2t) - \sin(2t)\right)$$

   Writing this as $R\cos(\omega t - \delta)$, we see that $R = \frac{2}{\sqrt{5}}$ and $\delta = \tan^{-1}(1/2) + \pi$, or

   $$y_p = \frac{2}{\sqrt{5}}\cos(2t - (\tan^{-1}(1/2) + \pi))$$

   For the next two problems, we'll take the shortcut formulas for the solution.

   (c) $y'' + 2y' + y = \cos(2t)$
   SOLUTION: Write $y'' + 2y' + y = \mathrm{e}^{2it}$, so $y_p = A\mathrm{e}^{2it}$. Now,

   $$A\mathrm{e}^{2it}(-4 + 2(2i) + 1) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-3+4i}$$

   Therefore, $y_p = R\cos(2t - \delta)$, where

   $$R = \frac{1}{|-3+4i|} = \frac{1}{\sqrt{5}} \qquad \delta = \tan^{-1}(-4/3) + \pi$$

   (d) $y'' + 2y' + 2y = \cos(t)$
   SOLUTION: Same idea as before, with $y_p = A\mathrm{e}^{it}$ and substitution gives:

   $$A\mathrm{e}^{it}(-1 + 2i + 2) = \mathrm{e}^{it} \quad \Rightarrow \quad A = \frac{1}{1+2i}$$

   Therefore, $y_p = R\cos(t - \delta)$, where

   $$R = \frac{1}{\sqrt{5}} \qquad \delta = \tan^{-1}(2)$$

2. Use the complexification technique to find the particular solution to $y'' + y = \mathrm{e}^{-t}\cos(t)$.
   There will be a bit of algebra involved, but not too bad- Write $g(t)$ as $\mathrm{e}^{(\alpha+\beta i)t}$

SOLUTION: First we complexify $g(t)$, so that $y'' + y = \mathrm{e}^{(-1+i)t}$. Then

$$y_p = A\mathrm{e}^{(-1+i)t}, \quad y_p' = A(-1+i)\mathrm{e}^{(-1+i)t}, \quad y_p'' = A(-1+i)^2\mathrm{e}^{(-1+i)t}$$

The DE becomes the following, where common factors have been taken out:

$$A\mathrm{e}^{(-1+i)t}((-1+i)^2+1) = \mathrm{e}^{(-1+i)t} \quad \Rightarrow \quad A = \frac{1}{1-2i}$$

We might ask if our shortcut formulas for R, δ still work- And they do! Consider that

$$A\mathrm{e}^{(-1+i)t} = \mathrm{e}^{-t}A\mathrm{e}^{it}$$

We can temporarily take out e^{-t}, compute $R\cos(\omega t - \delta)$, then remember to multiply the answer again by e^{-t}. In this case,

$$R = \frac{1}{\sqrt{5}}, \quad \delta = \tan^{-1}(-2), \quad y_p = \frac{1}{\sqrt{5}}\mathrm{e}^{-t}\cos(t - \tan^{-1}(-2))$$

3. If $F(t) = \frac{1}{\alpha+i\beta}\mathrm{e}^{i\omega t}$, then show (by direction computation) that the real part of F can be expressed as

$$\mathrm{Re}(F(t)) = \frac{1}{|\alpha+i\beta|}\cos(\omega t - \delta), \qquad \text{where } \delta = \arg(\alpha + i\beta)$$

And the imaginary part as:

$$\mathrm{Im}(F(t)) = \frac{1}{|\alpha+i\beta|}\cos(\omega t - \delta), \qquad \text{where } \delta = \arg(-\beta + i\alpha)$$

SOLUTION: These are direct computations.

Extra! The Complex Exponential and Integration

The complex exponential can also be used to help compute integrals involving the sine and cosine. As a simple example:

$$\int \cos(3t)\,dt$$

We can complexify the cosine so the integral becomes $\int \mathrm{e}^{3it}\,dt = \frac{1}{3i}\mathrm{e}^{3it}$, and now take the real part of the answer, which is $\frac{1}{3}\sin(3t) + C$.

We can do the same with some harder integral- Namely, an exponential times cosine or sine- We'll just complexify the integrand, then at the end take the real or imaginary part (respectively). Here's an example:

$$\int \mathrm{e}^{-t}\sin(2t)\,dt \quad \Rightarrow \quad \int \mathrm{e}^{(-1+2i)t}\,dt = \frac{1}{-1+2i}\mathrm{e}^{(-1+2i)t}$$

Of course, we need to be able to interpret and compute the answer. In this case, we want the imaginary part to finish:

$$\frac{1}{-1+2i}\mathrm{e}^{2it} = \frac{-1-2i}{5}(\cos(2t) + i\sin(2t))$$

Therefore,

$$\int \mathrm{e}^{-t}\sin(2t)\,dt = -\frac{2}{5}\cos(2t) - \frac{1}{5}\sin(2t) + C$$

Here are some worked examples for you to try yourself:

1. Use the complex exponential to integrate the following:

   (a) $\displaystyle\int e^{-2t}\cos(t)\,dt$
   SOLUTION: We'll take the real part of:

$$\int e^{-2t}e^{it}\,dt = \int e^{(-2+i)t}\,dt = \frac{1}{-2+i}e^{(-2+i)t}$$

   Expanding this,

$$\text{Real}\left(e^{-2t}\left(\frac{-2-i}{5}(\cos(t)+i\sin(t))\right)\right) = e^{-2t}\left(-\frac{2}{5}\cos(t)+\frac{1}{5}\sin(2t)\right)+C$$

   (b) $\displaystyle\int e^{t/2}\sin(3t)\,dt$
   SOLUTION: Same idea as before:

$$\int e^{t/2}e^{3it}\,dt = \int e^{\left(\frac{1}{2}+3i\right)t}\,dt = \frac{1}{\frac{1}{2}+3i}e^{\left(\frac{1}{2}+3i\right)t} = \frac{2}{1+6i}e^{t/2}(\cos(3t)+i\sin(3t))$$

   A little more simplification:

$$2e^{t/2}\left(\frac{-1-6i}{37}(\cos(3t)+i\sin(3t)\right)$$

   And we take the imaginary part, so that:

$$\int e^{t/2}\sin(3t)\,dt = e^{t/2}\left(-\frac{12}{37}\cos(3t)+\frac{2}{37}\sin(3t)\right)+C$$

   (c) $\int e^{-t}\cos(3t)\,dt$
   SOLUTION: Included here is some shortcut notation you might find useful when computing:

$$\int e^{(-1+3i)t}\,dt = \frac{1}{-1+3i}e^{(-1+3i)t} = \frac{e^{-t}}{10}(-1-3i)(c+is) = \frac{1}{10}((-c+3s)+i(-3c-s))$$

   Therefore, the integral is

$$-\frac{1}{10}e^{-t}\cos(3t)+\frac{3}{10}e^{-t}\sin(3t)+C$$