# Review of Linearization in Calculus

In Calculus I, the *linearization* of $y = f(x)$ at a point $(a, f(a))$ was defined to be the tangent line to $f$ at $a$:

$$L(x) = f(a) + f'(a)(x - a)$$

We are going to do something similar here. It is very difficult to analyze a general non-linear system of differential equations, so we will "linearize" the system at the equilibria, and we know how to analyze linear systems.

In Calculus III, we learn how to linearize $z = f(x, y)$ at a point $(a, b, f(a, b))$- we get the tangent plane:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

From this, you can guess what we do for $w = f(x, y, z)$ at $(a, b, c, f(a, b, c))$:

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and so on.

## Vector Fields and Linearization

Given a vector field $\mathbf{F}(x, y)$, we can also linearize the whole by linearizing each function. For example, if

$$\mathbf{F}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

Then the linearization at $x = a, y = b$ is given by:

$$\mathbf{L}(x, y) = \begin{bmatrix} f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ g(a, b) + g_x(a, b)(x - a) + g_y(a, b)(y - b) \end{bmatrix}$$

This is typically written in matrix-vector form:

$$\mathbf{L}(x, y) = \begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix} \begin{bmatrix} (x - a) \\ (y - b) \end{bmatrix} + \begin{bmatrix} f(a, b) \\ g(a, b) \end{bmatrix}$$

The matrix of first partial derivatives has a special name- It is the **Jacobian** matrix for our vector field- You might have heard of this matrix in Calc III since it comes up in optimization problems.

# Homework: Linearization and some review

1. Find all solutions to each system of equations.

   (a) $\begin{aligned} (2 + x)(y - x) &= 0 \\ (4 - x)(y + x) &= 0 \end{aligned}$ (b) $\begin{aligned} x + x^2 + y^2 &= 0 \\ y - xy &= 0 \end{aligned}$ (c) $\begin{aligned} x - x^2 - xy &= 0 \\ 3y - xy - 2y^2 &= 0 \end{aligned}$

2. Linearize each function about the given point.

   (a) $f(x) = x^2 + 2x + 1 + \cos(x)$ at $x = 0$
   
   (b) $f(x, y) = x - 2y + x^2 + e^{xy}$ at $x = 1, y = 0$.
   
   (c) $\mathbf{F}(x, y) = \begin{bmatrix} -x + y + 2xy \\ -4x - y + x^2 - y^2 \end{bmatrix}$ at $(x = 0, y = 0)$.
   
   (d) $\mathbf{F}(x, y) = \begin{bmatrix} (1 + x)\sin(y) \\ 1 - x - \cos(y) \end{bmatrix}$ at $(x = 0, y = 0)$.