# Local Linear Analysis of Nonlinear Autonomous DEs

In general, and in contrast to a *linear* system of differential equations, we will not be able to find nice analytic solutions to *nonlinear* systems.

How do we analyze solutions if we can't express them analytically? Several techniques exist- you can examine numerically generated solutions, or what we do is to do some graphical analysis and local linear analysis.

## Local Linearization

Local linear analysis is the process by which we analyze a nonlinear system of differential equations about its equilibrium solutions (also known as critical points or fixed points). This is like zooming into a graph and isolating our view of the DE so that we are just looking close by the equilibrium. Often, what we see looks just like a linear system.

Given a system of nonlinear autonomous DEs:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad \text{or} \qquad \mathbf{x}'(t) = F(\mathbf{x})$$

we first find the equilibrium solutions by setting the derivatives to zero, then solve simultaneously, the system:

$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}$$

Given an equilibrium, say $x = a, y = b$, the linearization of the system at the point $(a, b)$ is the following matrix, also called the **Jacobian matrix** of $F$.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

We can then use the Poincaré Diagram to determine the local behavior. We must use some caution in the case of centers and degenerate nodes, however. Because the linearization is an *approximation* of the true solution, the actual solutions are of a slightly perturbed system, and this has some implications.

For example, suppose we have a saddle so that $\lambda_1 > 0$ and $\lambda_2 < 0$. Then, no matter how close to zero these values are, there is a small window of adjustment that could be made; $\lambda_1 \pm \epsilon$ is still greater than zero and $\lambda_2 \pm \epsilon$ remains negative, so that the equilibrium remains a saddle.

However, consider $\lambda = \pm i$ for the linearized case, so that we predict that the equilibrium is a center. The problem is that very very slight perturbations might make $\lambda = \epsilon \pm i$, which destroys the center in favor of a spiral source or spiral sink- We would not be able to necessarily tell in the linearization.

For now, let's take a look at the process, and then we'll consider these special cases again later.

## Example 1: Competing Species

Suppose we have two populations that are competing for similar resources, like rabbits ($x(t)$) and hamsters ($y(t)$).

It seems reasonable to suppose that:

> In the absence of the other, each population is modeled by a population model with an environmental threshold (what we called the **logistic model**. Back in Chapter 2, that was (book's notation on p 81):
>
> $$y' = r(1 - y/k)y = ay - by^2$$
>
> so that $y = 0$ is unstable and $y = k$ is stable.

To simplify matters, we assume some constants:

$$\begin{aligned} x' &= x - x^2 \\ y' &= \tfrac{3}{4}y - y^2 \end{aligned}$$

Now, our second assumption will be:

> The rate of change of populations (down) will be proportional to the number of interactions between the populations. For example, if there are 3 rabbits and 2 hamsters, there are a total of 6 possible rabbit-hamster interactions possible.

Our equations become:

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= \tfrac{3}{4}y - y^2 - \tfrac{1}{2}xy \end{aligned}$$

The analysis proceeds by getting the **equilibria** (a.k.a. **critical points**):

From the first equation, either $x = 0$ or $x = -y + 1$:

- $x = 0$ in the second equation: $y\left(\frac{3}{4} - y\right) = 0$ so that $y = 0$ or $y = 3/4$.
- If $x = -y + 1$ in the first, then the second equation becomes:

$$y\left(\frac{3}{4} - y - \frac{1}{2}(-y+1)\right) = y\left(\frac{1}{4} - \frac{1}{2}y\right) = 0$$

Therefore, $y = 0$ (and $x = 0$, but we've counted that one), or $y = 1/2$ (then $x = 1/2$, too).

We have 4 equilibrium solutions:

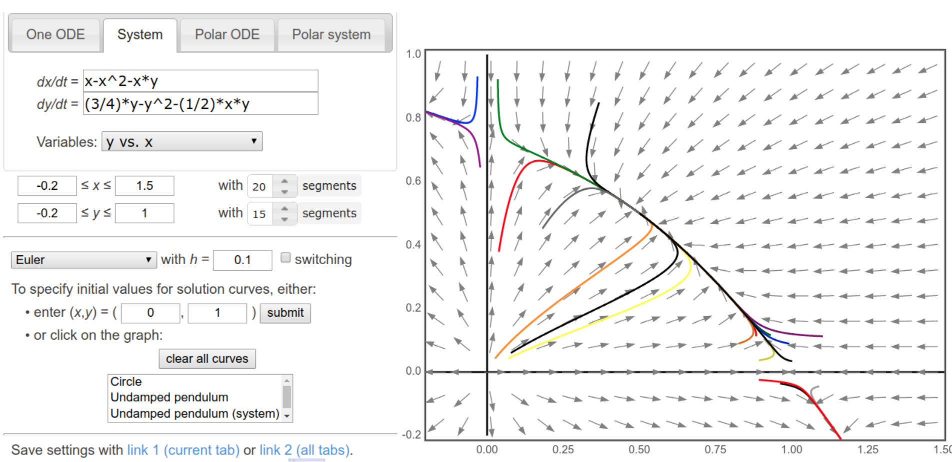
$$(0,0),\ (1,0),\ (0,3/4),\ (1/2,1/2)$$

Now we linearize the system about each equilibrium solution to determine its stability. First, the matrix of partial derivatives is:

$$\begin{bmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{bmatrix}$$

Evaluating this at each of the critical points (in order) gives us:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 0 & 0.25 \end{bmatrix} \quad \begin{bmatrix} 0.25 & 0 \\ -.375 & -.75 \end{bmatrix} \quad \begin{bmatrix} -.5 & -.5 \\ -.25 & -.5 \end{bmatrix}$$




Using the Poincaré Diagram, we see that the origin is indeed a SOURCE, the equilibria on the $x-$ and $y-$ axes are SADDLES, and the point of intersection of the two lines is a SINK. Putting these together, we can look at the direction field to examine the global behavior. From this we see that if both of the initial populations are not zero, the model predicts that all solutions will tend to the sink at $(1/2, 1/2)$- We might call this **peaceful coexistence**.

## Example 2:

We now repeat the analysis for a slightly different system:

$$\begin{array}{rcll} x' & = & x(1-x-y) & = x - x^2 - xy \\ y' & = & y(0.5 - 0.25y - 0.75x) & = 0.5y - 0.25y^2 - 0.75xy \end{array}$$

The critical points have changed slightly:

$$(0,0),\ (1,0),\ (0,2),\ (1/2,1/2)$$

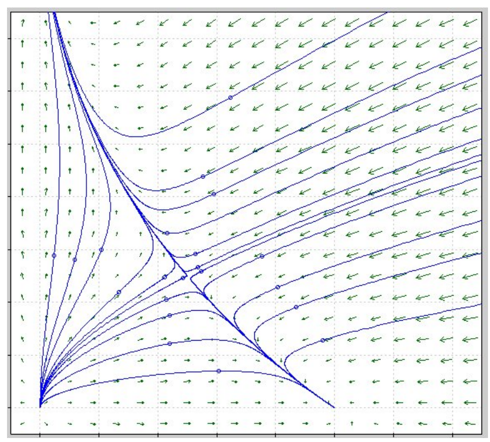
The matrix of partial derivatives is:

$$\left[\begin{array}{cc} 1-2x-y & -x \\ -.75y & 0.5-0.5y-0.75x \end{array}\right]$$

Evaluating this at each of the critical points (in order) gives us:

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 0.5 \end{array}\right] \quad \left[\begin{array}{cc} -1 & -1 \\ 0 & -0.25 \end{array}\right] \quad \left[\begin{array}{cc} -1 & 0 \\ -1.5 & -.5 \end{array}\right] \quad \left[\begin{array}{cc} -.5 & -.5 \\ -.375 & -.125 \end{array}\right]$$

Using the Poincaré Diagram, we see that the origin is still a SOURCE, the equilibria on the $x-$ and $y-$ axes are still SADDLES. The big difference is that the point of intersection of the two lines now produces a SADDLE. We will recall that a saddle point in UNSTABLE- this means that for just about any initial condition, one of the two species will die off.



## Example 3:

Consider the following system:

$$\begin{array}{rcll} x' & = & x - 0.5xy & = x(1-0.5y) \\ y' & = & -0.75y + 0.25xy & = y(-0.75 + 0.25x) \end{array}$$

Analyze the behavior of the solutions of the system by using local linearization.

SOLUTION: We always get equilibria first. In this case, from the first equation we get

- $x = 0$: From the second, we must have $y = 0$.
- $y = 2$: From the second, $x = 3$.

We have only two equilibria for this system, $(0,0)$ and $(3,2)$.

Now compute the "Jacobian matrix" of partial derivatives:

$$\left[\begin{array}{cc} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{cc} 1-0.5y & -0.5x \\ 0.25y & -0.75+0.25x \end{array}\right]$$

Linearizing about the two equilibrium gives (in order):

$$\begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}$$

In the first case, the trace is $1/4$ and the determinant is $-3/4$. By the Poincare diagram, the origin is a SADDLE.

At the point $(3, 2)$, the trace is 0 and the determinant is positive: We have a CENTER. We should check the direction field to verify our analysis.

In the figure below, we first show the $(x, y)$ plane, then in the next figure, we plot $x(t)$ and $y(t)$ versus $t$.

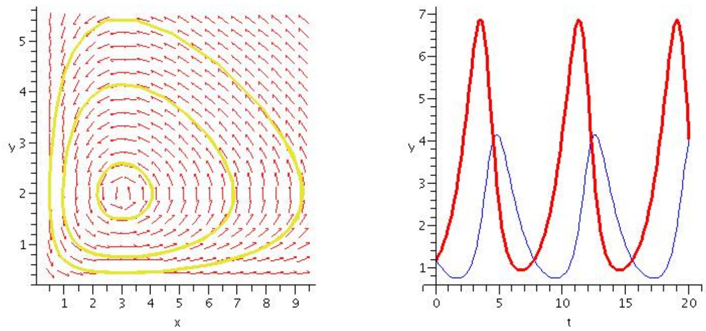



Figure 1: The direction field (top) and a plot of $x(t)$ and $y(t)$ as functions of time (bottom). They both reveal the existence of periodic solutions.

### What's next?

Back in the notes, we'll see some examples where linearization fails- That is, solutions to the globally nonlinear system are somehow significantly different than the approximation we make in the local linear system. As an point of information, consider the system of equations below. The origin is the only critical point, and yet we are getting some very interesting behavior. In fact, it is the cover of the Boyce and Diprima text!