## Case 3: One Real Eigenvalue, One Eigenvector

In the rare occurrence that you have one eigenvalue but two eigenvectors go to Case 1. For example, find the eigenvalues and eigenvectors to the identity matrix.

$$\begin{vmatrix} (1-\lambda) & 0 \\ 0 & (1-\lambda) \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = 1, 1$$

Now, solve the system for $\mathbf{v}$:

$$\begin{aligned} 0v_1 + 0v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned}$$

Both $v_1, v_2$ are free variables, so any vectors would work- We could use any two vectors (non-zero, not multiples of each other) to be our eigenvectors. Some like to use:

$$\mathbf{v} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And therefore, we have two eigenvectors, $[1, 0]^T$ and $[0, 1]^T$. This is not typical.

### Typical Case: A double eigenvalue, one eigenvector

Example: $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ In this case, $\lambda = 2, 2$ but

$$\begin{aligned} 0v_1 + 3v_2 &= 0 \\ 0v_1 + 0v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This one can be a little tricky, but there is an way to quickly get the solution if we have the intial conditions, $\mathbf{x}(0) = \mathbf{x}_0$. Then the solution to $\mathbf{x}' = A\mathbf{x}$ is given by:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

If we substitute this back into the DE, we will see that the following needs to hold:

$$(A - \lambda I)\mathbf{x}_0 = \mathbf{w}$$

**Example:**

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

We just computed the eigenvalues to be $\lambda = 2, 2$. To find the vector $\mathbf{w}$, we take:

$$\begin{aligned} (2-2)x_0 + 3y_0 &= w_1 \\ 0x_0 + (2-2)y_0 &= w_2 \end{aligned} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 3y_0 \\ 0 \end{bmatrix}$$

The full solution is then:

$$\mathbf{x}(t) = \mathrm{e}^{2t} \left( \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 3y_0 \\ 0 \end{bmatrix} \right)$$

**Example:**

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x}$$

The trace is 0 and the determinant is 0. Therefore, $\lambda = 0$ is the only eigenvalue. If there are no initial conditions, assume they are $(x_0, y_0)$ as the last example. Then our vector $\mathbf{w}$ is computed as:

$$\begin{aligned} (4-0)x_0 - 2y_0 &= w_1 \\ 8x_0 - (4-0)y_0 &= w_2 \end{aligned} \qquad \mathbf{w} = \begin{bmatrix} 4x_0 - 2y_0 \\ 8x_0 - 4y_0 \end{bmatrix}$$

The solution is (in several forms):

$$\mathbf{x}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 4x_0 - 2y_0 \\ 8x_0 - 4y_0 \end{bmatrix}$$

We'll note that this is just a straight line in the $(x_1, x_2)$ plane.

## Summary

To solve $\mathbf{x}' = A\mathbf{x}$, find the trace, determinant and discriminant. The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on $\Delta$:

- Real $\lambda_1, \lambda_2$ give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

  $$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

  $$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector $\mathbf{v}$ (not used directly).

  Use the initial condition, $\mathbf{x}_0 = (x_0, y_0)$ and the vector $\mathbf{w}$ so that

  $$\begin{aligned} (a-\lambda)x_0 + by_0 &= w_1 \\ cx_0 + (d-\lambda)y_0 &= w_2 \end{aligned} \quad \Leftrightarrow \quad (A - \lambda I)\mathbf{x}_0 = \mathbf{w}$$

  The solution is then

  $$\mathbf{x}(t) = \mathrm{e}^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

Further, be able to plot solutions for sinks, sources, saddles, spirals and centers. You do not have to plot solutions for this third case, which represents "degenerate" cases.