Poincare Classification

- Earlier, the autonomous DE was: $y' = f(y)$, and one goal was to identify and classify the equilibria (we looked at plot of y v. y').
- Goal today: Classify the origin in $\mathbf{x}' = A\mathbf{x}$

  So far, we've seen sinks, saddles, sources, centers, spiral sinks and spiral sources. Is there a way to organize these?

Recall

Solve $\mathbf{x}' = A\mathbf{x}$:

Find the eigenvalues/eigenvectors (through the trace, det and discriminant)

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

Depends on Δ:

- Real λ_1, λ_2 give two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Real}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Imag}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector. Given $\mathbf{x}_0$, find $\mathbf{w}$, where $\begin{aligned}(a-\lambda)x_0 + by_0 &= w_1 \\ cx_0 + (d-\lambda)y_0 &= w_2\end{aligned}$.

  Then:

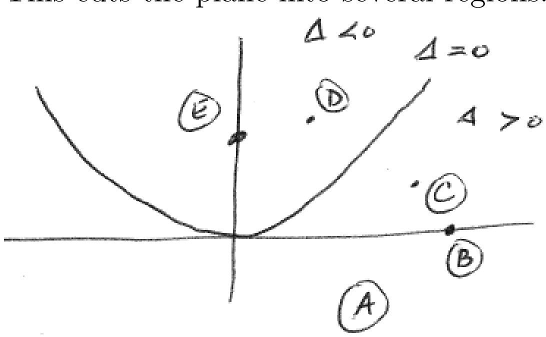
$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\left(\mathbf{x}_0 + t\mathbf{w}\right)$$

So the type of solution depends on Δ, and in particular, where $\Delta = 0$:

$$\Delta = 0 \quad \Rightarrow \quad 0 = (\mathrm{Tr}(A))^2 - 4\det(A)$$

This is a parabola in the $(\mathrm{Tr}(A), \det(A))$ coordinate system.

This cuts the plane into several regions:



1. In Region A, the det is negative, and $\Delta > 0$. In examining λ,

$$\frac{\operatorname{Tr}(A) \pm \sqrt{\operatorname{Tr}(A)^2 - 4\det(A)}}{2}$$

With det< 0, that means $(\operatorname{Tr}(A))^2 < (\operatorname{Tr}(A))^2 - 4\det(A)$, so λ will ALWAYS be mixed in sign (pos, neg), and we will always get a SADDLE.

2. In Region B, the determinant is zero, so we always have a line of fixed points, and our eigenvalues are

$$\lambda = 0, \operatorname{Tr}(A)$$

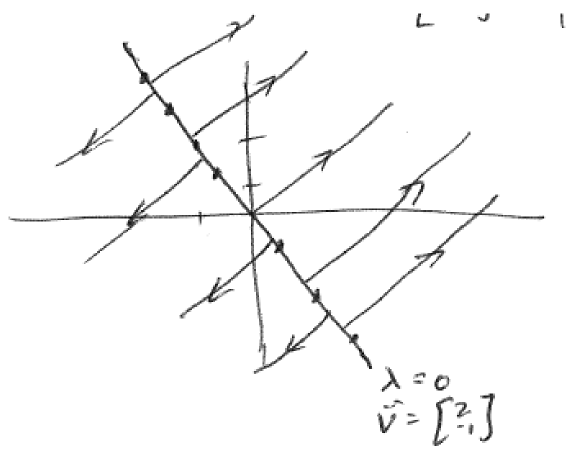
With the trace being positive, the trajectory is away from the line of fixed points. Here's a quick example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} \lambda = 0 \\ \mathbf{v} = (2, -1) \end{array} \qquad \begin{array}{c} \lambda = 5 \\ \mathbf{v} = (1, 2) \end{array}$$

The general solution is

$$\mathbf{Y}(t) = C_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

See the graph below.



3. For Region C, the determinant and trace are positive, and the discriminant is positive. Therefore,

$$\operatorname{Tr}(A) > \sqrt{\operatorname{Tr}(A)^2 - 4\det(A)} \quad \Rightarrow \quad \lambda_{1,2} > 0$$

Therefore, we have a SOURCE.

4. On the curve, we have a DEGENERATE source.

5. Inside the parabola in region D, the discriminant is negative,

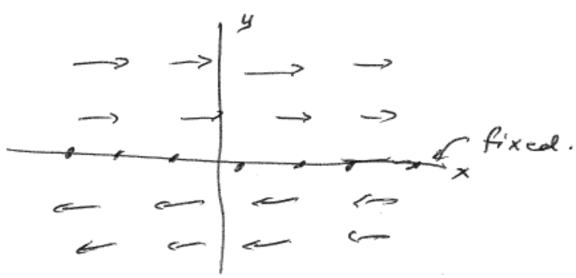
$$\lambda_{1,2} = \frac{\mathrm{Tr}(A)}{2} \pm \frac{\sqrt{-\Delta}}{2}\, i$$

Therefore, we have a SPIRAL SOURCE.

6. We might consider one last special case- The origin. In that case, both the trace and determinant are zero, so $\lambda = 0, 0$. There are actually two cases to think about, depending on the number of eigenvectors. Compare:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For A_1, we see that $x(t) = C_1, y(t) = C_2$, and the entire plane is fixed. For A_2, we see that $y(t) = C_1$, then $x(t) = C_1 t + C_2$. The $x-$axis is fixed. Above the $x-$axis, solutions are horizontal lines moving to the right, below we have horizontal lines moving to the left.



7. If the determinant is positive, and the trace is zero, then

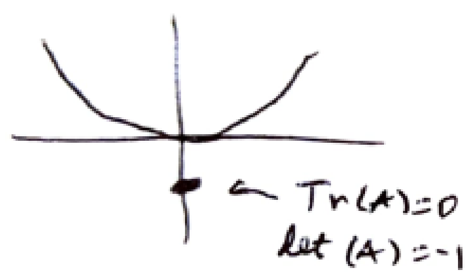
$$\lambda = \pm \frac{\sqrt{-4\det(A)}}{2} = \pm \beta i$$

which corresponds to a CENTER. The circular frequency is β, the period is $2\pi/\beta$.

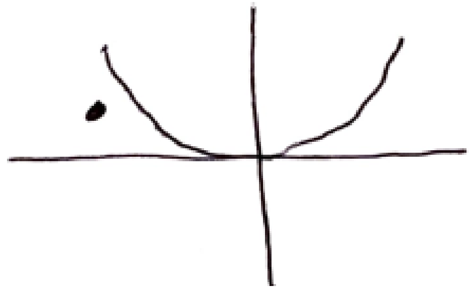
8. The other regions are determined in a similar manner...

Examples: Using the Poincaré Diagram

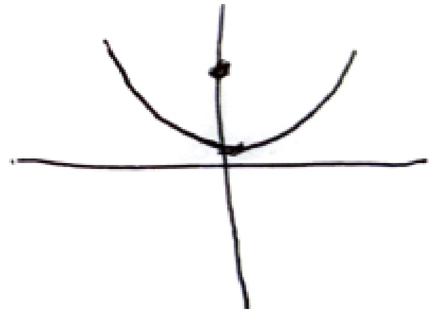
- Suppose $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$. Then the trace is 0 and the determinant is -1. Where are we on the diagram? In the diagram, we are in the region designated as *saddle*. Therefore the origin (the only equilibrium point) is classified as a saddle.



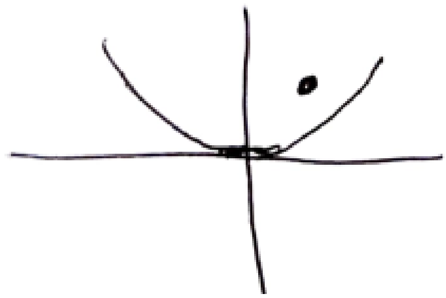
- Suppose $\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 0 & -1/4 \end{bmatrix} \mathbf{x}$. Then the trace is $-5/4$ and the determinant is $1/4$ and the discriminant is $9/16$. Where are we on the diagram? The origin (the only equilibrium point) is classified as a sink.



- Suppose $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \mathbf{x}$. Then the trace is 0 and the determinant is 9 and the discriminant is -36. Where are we on the diagram? The origin (the only equilibrium point) is classified as a center.



- Suppose $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}$. Then the trace is 2 and the determinant is 5 and the discriminant is -16. Where are we on the diagram? The origin (the only equilibrium point) is classified as a spiral source.



Varying a Parameter

Suppose that the matrix has a parameter that varies. We can use the diagram to examine how changing the parameter changes the nature (or classification) of the origin.

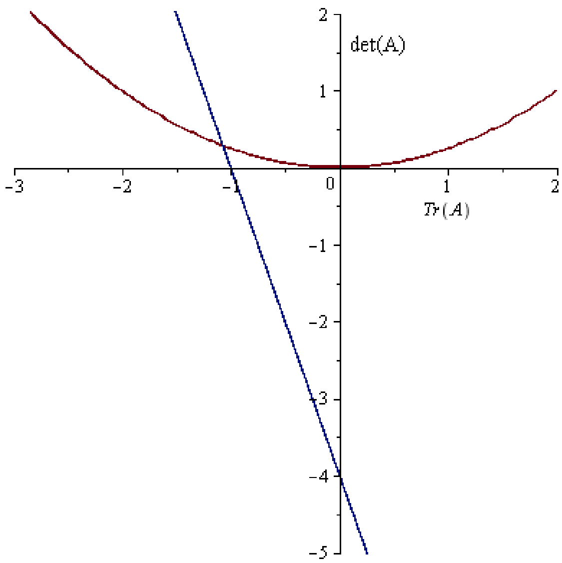
For example, suppose the matrix, trace, determinant and discriminant are as shown:

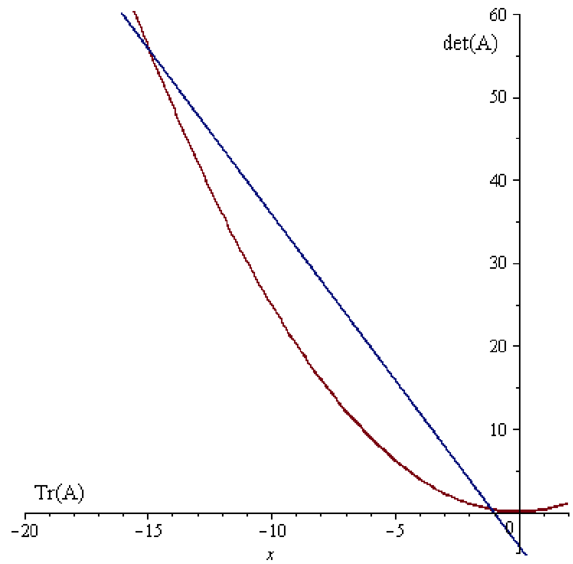
$$A = \begin{bmatrix} a & a \\ 3 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} \mathrm{Tr}(A) = a - 1 \\ \det(A) = -a - 3a = -4a \\ \Delta = (a-1)^2 - 4(-4a) = (a-1)^2 + 16a = (a-7)^2 - 48 \end{array}$$

If you can do it fairly easily, it may be best to translate the trace and det as parametric functions of a. Translating the trace as x, the determinant as y and setting $a = t$ so things look familiar, we have:

$$\begin{array}{c} x = t - 1 \\ y = -4t \end{array} \quad \Rightarrow \quad t = x + 1 \Rightarrow y = -4(x+1)$$

If we overlay this line on top of the parabola in the $(\mathrm{Tr}(A), \det(A))$ plane, we get the following graphs- The one on the left is a closeup around the origin, the second is a more global view (see the scalings).





We want to focus on the line segment, since, as our parameter a changes, we slide along the line. For example, which value of a will put us at $(0, -4)$? If $a = 1$, we can check that the trace is 0 and determinant is -4. For the rest of the algebra, it is useful to break out a tool from Calculus, where we had to find where expressions were positive/negative (a sign chart analysis). The expressions we're checking are listed to the left above the number line we're using for our parameter (in this case, a). The number line is broken up where the expressions are equal to zero. In this case, we see $a = 1$, $a = 0$ and $a = -7 \pm \sqrt{48}$ are the four zeros. This breaks the number line into 5 test regions. We simply test each each expression in each region to see where the expressions are positive or negative. For brevity, let $p_1 = -7 + \sqrt{48}$ and $p_2 = -7 - \sqrt{48}$.

$\mathrm{Tr}(A) = a - 1$	$-$	$-$	$-$	$-$	$+$
$\det(A) = -4a$	$+$	$+$	$+$	$-$	$-$
$\Delta = (a-7)^2 - 48$	$+$	$-$	$+$	$+$	$+$
	$a < p_2$	$p_2 < a < p_1$	$p_1 < a < 0$	$0 < a < 1$	$a > 1$

We can read the classification off the graph, or on the sign chart:

- If $a > 0$ then we have a saddle (nothing changes at $a = 1$).
- At $a = 0$, we have a line of attracting fixed points.
- For $p_1 < a < 0$, we have a sink.
- At $a = p_1$, we have a degenerate sink.
- For $p_2 < a < p_1$, we have a spiral sink.
- For $a = p_2$, we have a degenerate sink.
- For $a < p_2$, we have a sink.

Homework: Poincare Classification

1. The matrices below represent the matrix A in a linear system, $\mathbf{x}' = A\mathbf{x}$. Classify the equilibrium (the origin) using the Poincaré Classification:

   (a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$

   (c) $\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$

   (d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

2. Discuss how the classification of the origin changes with α, given the matrix below.

   (a) $\begin{bmatrix} \alpha & -1 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \alpha & \alpha \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} \alpha & 1 \\ \alpha & \alpha \end{bmatrix}$

3. Suppose we are given $\mathbf{x}' = A\mathbf{x}$, and we compute the eigenvalues and eigenvectors (below). Draw a sketch of the phase plane in each case.

   (a) $\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (b) $\lambda_1 = -1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = -2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (c) $\lambda_1 = 1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with $\lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4. Each matrix below represents the matrix A in the system $\mathbf{x}' = A\mathbf{x}$. For each, (i) Use the Poincaré classification, (ii) write down the general solution using eigenvalues/eigenvectors, and (iii) draw a sketch of the phase plane (the (x_1, x_2) plane).

   (a) $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$