Review Questions, Exam 1, Math 244

These questions are presented to give you an idea of the variety and style of question that will be on the exam. It is not meant to be exhaustive, so be sure that you understand the homework problems and quizzes.

1. Solve. If there is an initial condition, solve the IVP.

   (a) $y' = \dfrac{x^2 - 2y}{x}$

   (b) $y' = \dfrac{2t + y}{3 + 3y^2 - t} \qquad y(0) = 0.$

   (c) $y' = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$

   (d) $y' = 2\cos(3t) \quad y(0) = 2$

   (e) $y' - \frac{1}{2}y = 0 \qquad y(0) = 200.$

   (f) $y' = (1 - 2x)y^2 \qquad y(0) = -1/6.$

   (g) $y' - \frac{1}{2}y = \mathrm{e}^{2t} \qquad y(0) = 1$

   (h) $y' = y(3 - y)$

   (i) $\sin(2t)\,dt + \cos(3y)\,dy = 0$

   (j) $y' = xy^2$

   (k) $y' + 2y = g(t)$, $y(0) = 0$ and

   $$g(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$$

2. Substitution:

   (a) Show that the DE becomes separable with the change of variables: $v = y/x$: $(x+y)\,dx - (x-y)\,dy = 0$.

   (b) Show that the DE becomes linear in v with the change of variables: $v = y^2$: $\dfrac{dy}{dx} - \dfrac{3}{2x}y = \dfrac{2x}{y}$

   (c) Let $p(y) = y'$, and rewrite the DE below in terms of p, y alone. $yy'' = (y')^2$ (Hint: Let $p(y) = y'$, and see if you can get the equation in terms of p and y).

3. True or False, and explain:

   (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

   (b) Let $\frac{dy}{dt} = 1 + y^2$. The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all t. (If true, say why. If False, solve the DE).

   (c) If $y' = \cos(y)$, then the solutions are periodic.

   (d) All autonomous equations are separable.

   (e) All separable equations are exact.

4. Let $y' = y^{1/3}$, $y(0) = 0$. Find two solutions to the IVP. Does this violate the Existence and Uniqueness Theorem?

5. Let $(t+1)(t-2)y' + 3ty = \mathrm{e}^t$, with $y(1) = 2$. What is the largest interval on which we can guarantee the existence of a unique solution? (Hint: Do NOT solve the IVP).

6. If $\frac{dy}{dx} = \frac{x-3}{(x-1)(y-2)}$, then

   (a) find the areas of the plane that our initial value must avoid in order to (be explicit in your computations!), and

   (b) if $y(0) = 1$, is it possible to predict the interval on which the solution exists?

7. Find a constant a and a function $g(t)$ if $y(t)$ is to be a solution to the DE: $y' + ay = g(t)$.

   (a) $y(t) = t - 3 + C\mathrm{e}^{3t}$

   (b) $y(t) = 2\sin(3t) + C\mathrm{e}^{-2t}$

8. Suppose we want to construct a population model using our logistic model. Assuming the initial population growth rate is 0.1, and the environmental threshold (or carrying capacity) is 100, write down the model. Using your model, now assume there is a continuous"harvest" (of k units per time period). How does that effect the model- In particular, is there a critical value of k over which the population will be extinct? If so, find it.

9. Construct an autonomous differential equation with three equilibria: $y = -2, y = 0, y = 1$, and the two outer equilibria are stable, the inner equilibrium is unstable. (HINT: Start with a sketch in the (y, y') plane).

10. Suppose we have a tank that contains M gallons of water, in which there is Q_0 pounds of salt. Liquid is pouring into the tank at a concentration of r pounds per gallon, and at a rate of γ gallons per minute. The well mixed solution leaves the tank at a rate of γ gallons per minute.

    Write the initial value problem that describes the amount of salt in the tank at time t, and solve:

11. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on Q_0? Does this make sense?

12. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.

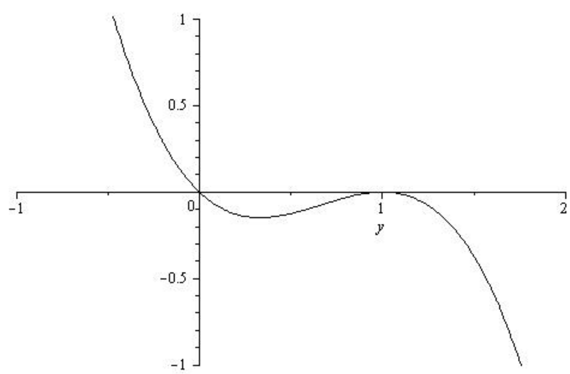
13. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time t. In the (t, y) plane, draw several solution curves (Hint: This is an autonomous DE).

14. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

15. Show that the IVP $xy' = y - 1$, $y(0) = 2$ has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution. You might start by actually trying to solve it).

16. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is 20^2), and that one year later, the population is 625 (which is 25^2). Determine the time in which the population reaches 10000 (which is 100^2).

17. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.

    (a) Find and classify the equilibrium.

    (b) Find intervals (in y) on which $y(t)$ is concave up.

    (c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down.

    (d) Find an appropriate polynomial for $F(y)$.

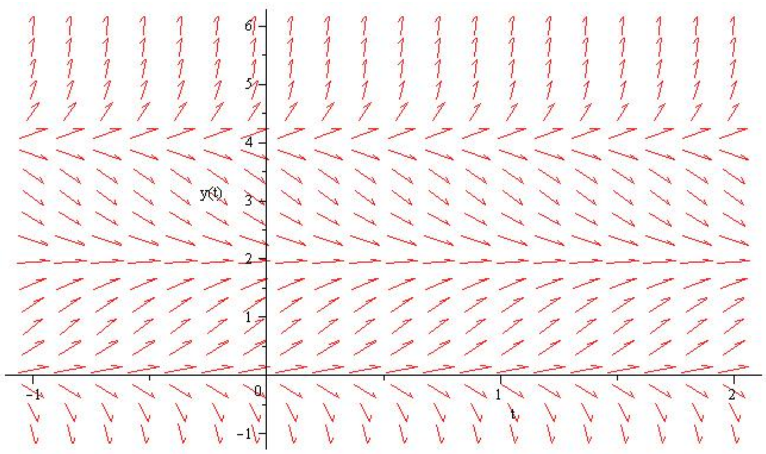
18. Evaluate the following integrals:

$$\int x^3 e^{2x}\,dx \qquad \int \frac{x}{(x-1)(2-x)}\,dx \qquad e^{-3\int dt/t}$$

19. Consider the DE: $y\,dx + (2x - ye^y)\,dy = 0$. Show that the equation is not exact, but becomes exact if you assume there is an integrating factor in terms of y alone (Hint: Find the integrating factor first).

20. Newton's Law of Cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the environment (which we assume is some constant). Write down a differential equation which represents this statement, then find the general solution.

21. Given the direction field below, find a differential equation that is consistent with it.



22. Consider the direction field below, and answer the following questions:

    (a) Is the DE possibly of the form $y' = f(t)$?

    (b) Is the DE possible of the form $y' = f(y)$?

    (c) Is there an equilibrium solution? (If so, state it):

    (d) Draw the solution corresponding to $y(-1) = 1$.