Study Guide: Exam 1, Math 244

The exam covers material from Chapters 1 and 2 (up to 2.6), and will be 50 minutes in length. You may not use the text, notes, colleagues or a calculator.

Because a differential equation defines a function (the solution), there are several ways of getting insight into the solution- Graphically, Algebraically, and Numerically. In Chapters 1 and 2, we get a little of the first and third, and a lot of the second.

In summary, the first exam is all about understanding (and solving) first order differential equations: $y' = f(t, y)$.

Vocabulary

- You should know what these terms mean (mostly from Ch 1)

  differential equation, ordinary differential equation, partial differential equation, order of a differential equation, linear differential equation, equilibrium solution, isocline, direction field

- Understand what it means for a **given** function to be a solution to a DE (like questions from 1.3).
- Be able to identify the following types of DEs: Linear, separable, autonomous. Know that these categories may overlap (a DE might be linear, separable, and autonomous, for example).

The Existence and Uniqueness Theorem

1. Linear: $y' + p(t)y = g(t)$ at (t_0, y_0):

   If p, g are continuous on an interval I that contains t_0, then there exists a unique solution to the initial value problem and that solution is valid for all $t \in I$.

2. General Case: $y' = f(t, y)$, (t_0, y_0):

   Let the functions f and f_y be continuous in some open rectangle R containing the point (t_0, y_0). Then there exists an interval about t_0, $(t_0 - h, t_0 + h)$ contained in R for which a unique solution to the IVP exists.

   Side Remark 1: To determine the full time interval, we must solve the DE. The theorem only says that there is a (perhaps) tiny interval about t_0.

   Side Remark 2: We broke out the theorem in class into two components (existence and uniqueness). You can use either the theorem there or as it stated above.

Graphical Analysis

1. Be able to use a direction field to analyze the behavior of solutions to general first order equations. Be able to construct simple direction fields using isoclines.

2. Special Case: **Autonomous DEs:** The main idea here is to be able to graph the phase plot, $y' = f(y)$ in the (y, y') plane and be able to translate the information from this graph to the direction field, the (t, y) plane.

   Here is a summary of that information:

In Phase Diagram:	In Direction Field:
y intercepts	Equilibrium Solutions
$+$ to $-$ crossing	Stable Equilibrium
$-$ to $+$ crossing	Unstable Equilibrium
$y' > 0$	y increasing
$y' < 0$	y decreasing
y' and df/dy same sign	y is concave up
y' and df/dy mixed	y is concave down

Recall that we also looked at a theorem about determining the stability of an equilibrium solution using the sign of df/dy, and determining a formula for y'' given $y' = f(y)$.

Analytic Solutions

- Linear: $y' + p(t)y = g(t)$. Use the integrating factor: $e^{\int p(t)\,dt}$
- Separable: $y' = f(y)g(t)$. Separate variables: $(1/f(y))\,dy = g(t)\,dt$
- Solve by substitution:
  - Homogeneous: $\frac{dy}{dx} = F(y/x)$. Substitute $v = y/x$ (and get the expression for dv/dx as well).
  - Bernoulli: $y' + p(t)y = g(t)y^n$ Divide by y^n, let $w = y^{1-n}$ and it becomes linear.

  NOTE: I'll give the substitution for these- For example, "If $v = 1/y^2$, show the DE becomes linear."

- Exact: $M(x,y) + N(x,y)\frac{dy}{dx}$, where $N_x = M_y$.

  We should recognize that we're comparing this to the total derivative from Calculus,

  $$f_x(x,y) + f_y(x,y)\frac{dy}{dx} = 0$$

  so that $M_y = N_x = f_{xy}$ for the unknown function f. To find f, we can find the potential for which $\nabla f = \langle M, N \rangle$. That is, we can take $f(x,y) = \int M(x,y)\,dx + G(y)$. Integrate w/r to x. Check that $f_y = N(x,y)$, and add a function of y if necessary.

  NOTE: I'll give an integrating factor for exact equations, if necessary. You should be able to derive equations that define the integrating factor, as done in class and on pages 98-99. That is, if you look in the book, see if you can figure out how Equation 27 on pg. 99 was derived.

Models

Be familiar with (be able to construct) the following models:

Exponential growth, Logistic growth, Free fall, Newton's Law of Cooling, Tank Mixing, and compound interest (with continuous compounding).

For growth problems, be able to solve for the appropriate constant(s) when given doubling time. For any physics problems, values of constants (like g) would be provided.