Overview of Complex Numbers

1 Initial Definitions

Definition 1 *The* **complex number** z *is defined as:* $z = a + bi$*, where* a*,* b *are real numbers and* $i = \sqrt{-1}$.

General notes about $z = a + bi$

- Engineers typically use j instead of i.
- Examples of complex numbers: $5 + 2i, \quad 3 - \sqrt{2}i, \quad 3, \quad -5i$
- Powers of i are cyclic: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$ and so on. Notice that the cycle is: $i, -1, -i, 1$, then it repeats.
- All real numbers are also complex (by taking $b = 0$), so the set of real numbers is a subset of the complex numbers.

We can split up a complex number by using **the real part** and **the imaginary part** of the number z:

Definition: The **real part** of $z = a + bi$ is a, or in notation we write: $\mathrm{Re}(z) = \mathrm{Re}(a+bi) = a$
The **imaginary part** of $a + bi$ is b, or in notation we write: $\mathrm{Im}(z) = \mathrm{Im}(a + bi) = b$

2 Visualizing Complex Numbers

To visualize a complex number, we use the complex plane $\mathbb{C}$, where the horizontal (or x-) axis is for the real part, and the vertical axis is for the imaginary part. That is, $a + bi$ is plotted as the point (a, b).

In Figure 1, we can see that it is also possible to represent the point $a + bi$, or (a, b) in **polar form**, by computing its modulus (or size) r, and angle (or argument) θ as:

$$r = |z| = \sqrt{a^2 + b^2} \qquad \theta = \arg(z)$$

Once we do that, we can write:

$$z = a + bi = r(\cos(\theta) + i\sin(\theta))$$

We have to be a bit careful defining θ. For example, just adding a multiple of 2π will yield an equivalent number for θ. Typically, θ is defined to be the 4-quadrant "inverse tangent"[1] that returns $-\pi < \theta \leq \pi$.

That is, formally we can define the argument as the following, which looks more complicated than it actually is. Highly recommended: Draw the point $a + ib$ in the complex plane. Then θ is given by:

[1] For example, in Maple this special angle is computed as `arctan(b,a)`, and in Matlab the command is `atan2(b,a)`.

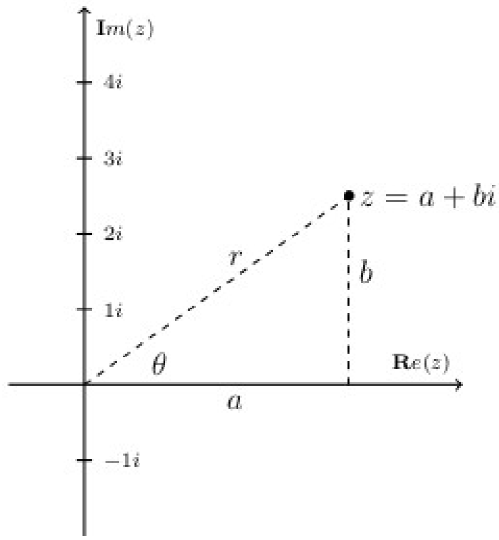

Figure 1: Visualizing $z = a + bi$ in the complex plane. Shown are the modulus (or length) r and the argument (or angle) θ.

- If (a, b) is in Quadrant I or IV, $\theta = \tan^{-1}(b/a)$.
- If (a, b) is on the upper vertical axis ($a = 0$), then $\theta = \pi/2$.
- If (a, b) is on the lower vertical axis ($a = 0$), then $\theta = -\pi/2$.
- If (a, b) is in Quadrant II or III, add π: $\theta = \tan^{-1}(b/a) + \pi$.
- At the origin, θ is said to be undefined.

Examples

Find the modulus r and argument θ for the following numbers, then express z in polar form:

- $z = -3$:

  SOLUTION: $r = 3$ and $\theta = \pi$ so $z = 3(\cos(\pi) + i\sin(\pi))$

- $z = 2i$:

  SOLUTION: $r = 2$ and $\theta = \pi/2$ so $z = 2(\cos(\pi/2) + i\sin(\pi/2))$

- $z = -1 + i$:

  SOLUTION: $r = \sqrt{2}$ and $\theta = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ so

$$z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

- $z = -3 - 2i$ (Numerical approx from Calculator OK):
  SOLUTION: $r = \sqrt{14}$ and $\theta = \tan^{-1}(2/3) - \pi \approx 0.588 - \pi \approx -2.55$ rad, or

$$z = \sqrt{14}\,(\cos(-2.55) + i\sin(-2.55)) = \sqrt{14}(\cos(2.55) - i\sin(2.55))$$

  Note to readers: We used the "even" symmetry of the cosine and the "odd" symmetry of the sine to do the simplification:

$$\cos(-x) = \cos(x) \qquad \text{and} \qquad \sin(-x) = -\sin(x)$$

3 Operations on Complex Numbers

3.1 The Conjugate of a Complex Number

If $z = a + bi$ is a complex number, then its *conjugate*, denoted by $\bar{z}$ is $a - bi$. For example,

$$z = 3 + 5i \Rightarrow \bar{z} = 3 - 5i \qquad z = i \Rightarrow \bar{z} = -i \qquad z = 3 \Rightarrow \bar{z} = 3$$

Graphically, the conjugate of a complex number is it's mirror image across the horizontal axis. If z has magnitude r and argument θ, then $\bar{z}$ has the same magnitude with a negative argument.

Example

If $z = 3(\cos(\pi/2) + i\sin(\pi/2))$, find the conjugate $\bar{z}$:

$$\bar{z} = 3(\cos(-\pi/2) + i\sin(-\pi/2)) = 3(\cos(\pi/2) - i\sin(\pi/2))$$

3.2 Addition/Subtraction, Multiplication/Division

To add (or subtract) two complex numbers, add (or subtract) the real parts and the imaginary parts separately. This is like adding polynomials (with i in place of x):

$$(a + bi) \pm (c + di) = (a + c) \pm (b + d)i$$

To multiply, expand it as if you were multiplying polynomials, with i in place of x:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

and simplify using $i^2 = -1$. A special product is often computed- A complex number with its conjugate:

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2 = |z|^2$$

Division by complex numbers $\frac{z}{w}$, is defined by translating it to real number division by rationalizing the denominator- multiply top and bottom by the conjugate of the denominator:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Example:

$$\frac{1+2i}{3-5i} = \frac{(1+2i)(3+5i)}{(3-5i)(3+5i)} = \frac{(1+2i)(3+5i)}{3^2+5^2} = \frac{-7}{34} + \frac{11}{34}i$$

4 The Polar Form of Complex Numbers

The polar form of a complex number,

$$z = r\cos(\theta) + ir\sin(\theta)$$

has a beautiful counterpart using the complex exponential function, $\mathrm{e}^{i\theta}$. First, we'll define it using Euler's formula (although it is possible to *prove* Euler's formula).

Definition (Euler's Formula): $\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$.

> We can now express the polar form of a complex number slightly differently:
>
> $$z = r\mathrm{e}^{i\theta} \quad \text{where} \quad r = |z| = \sqrt{a^2+b^2} \qquad \theta = \arg(z)$$

An important note about this expression: The rules of exponentiation still apply in the complex case. For example,

$$\mathrm{e}^{a+ib} = \mathrm{e}^a\mathrm{e}^{ib} \qquad \text{and} \qquad \mathrm{e}^{i\theta}\mathrm{e}^{i\beta} = \mathrm{e}^{(\theta+\beta)i} \qquad \text{and} \qquad \left(\mathrm{e}^{i\theta}\right)^n = \mathrm{e}^{in\theta}$$

Furthermore, in the next section, we'll look at the logarithm.

Examples

Given the complex number in $a+bi$ form, give the polar form, and vice-versa:

1. $z = 2i$

   SOLUTION: Since $r = 2$ and $\theta = \pi/2$, $z = 2\mathrm{e}^{i\pi/2}$

2. $z = 2\mathrm{e}^{-i\pi/3}$

   SOLUTION: We recall that $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$, so

   $$z = 2(\cos(-\pi/3) + i\sin(-\pi/3)) = 2(\cos(\pi/3) - i\sin(\pi/3)) = 1 - \sqrt{3}\,i$$

5 Exponentials and Logs

The logarithm of a complex number is easy to compute if the number is in polar form. We use the normal rule of logs: $\ln(ab) = \ln(a) + \ln(b)$, or in the case of polar form:

$$\ln(a + bi) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta$$

Where we leave the last step as intuitively clear, but we don't prove it here (we have to be careful about the choice of θ as described earlier).

The logarithm of zero is left undefined (as in the real case). However, we can now compute things like the log of a negative number!

$$\ln(-1) = \ln\left(1 \cdot e^{i\pi}\right) = i\pi \qquad \text{or the log of } i: \quad \ln(i) = \ln(1) + \frac{\pi}{2}i = \frac{\pi}{2}i$$

To exponentiate a number, we convert it to multiplication (a trick we used in Calculus when dealing with things like x^x):

$$a^b = e^{b\ln(a)}$$

Examples of Exponentiation

- $2^i = e^{i\ln(2)} = \cos(\ln(2)) + i\sin(\ln(2))$
- $\sqrt{1+i} = (1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = (2^{1/4})e^{i\pi/8}$
- $i^i = e^{i\ln(i)} = e^{i(i\pi/2)} = e^{-\pi/2}$

6 Real Polynomials and Complex Numbers

If $ax^2 + bx + c = 0$, then the solutions come from the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the past, we only took real roots. Now we can use complex roots. For example, the roots of $x^2 + 1 = 0$ are $x = i$ and $x = -i$.

Check:

$$(x - i)(x + i) = x^2 + xi - xi - i^2 = x^2 + 1$$

Some facts about polynomials when we allow complex roots:

1. An n^{th} degree polynomial can always be factored into n roots. (Unlike if we only have real roots!) This is the *Fundamental Theorem of Algebra*.
2. If $a+bi$ is a root to a real polynomial, then $a-bi$ must also be a root. This is sometimes referred to as "roots must come in conjugate pairs".

7 Exercises

1. Suppose the roots to a cubic polynomial are $a = 3$, $b = 1-2i$ and $c = 1+2i$. Compute $(x-a)(x-b)(x-c)$.

2. Find the roots to $x^2 - 2x + 10$. Write them in polar form.

3. Show that:
$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2} \qquad \mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$$

4. For the following, let $z_1 = -3+2i$, $z_2 = -4i$
   (a) Compute $z_1\bar{z}_2$, z_2/z_1
   (b) Write z_1 and z_2 in polar form.

5. In each problem, rewrite each of the following in the form $a + bi$:
   (a) e^{1+2i}
   (b) e^{2-3i}
   (c) $\mathrm{e}^{i\pi}$
   (d) 2^{1-i}
   (e) $\mathrm{e}^{2-\frac{\pi}{2}i}$
   (f) π^i

6. For fun, compute the logarithm of each number:
   (a) $\ln(-3)$
   (b) $\ln(-1+i)$
   (c) $\ln(2\mathrm{e}^{3i})$