# Complexification

Now that we've worked a bit with complex numbers, we might take advantage of the fact that often, rather than dealing with $\sin(x)$ and $\cos(x)$, it might be a lot easier to work with the complex exponential function.

In things like integration and as a forcing function for the linear DE, it is especially convenient. For example, if we have a linear operator and we need to apply it to $\cos(\omega t)$, why not apply it to the exponential function instead?

Here's a specific example so we can see what's going on. Suppose we want to integrate $\cos(t)$ (which is a linear operation):

$$\int \cos(t)\,dt \quad \Rightarrow \quad \int \cos(t)\,dt + i\int \sin(t)\,dt = \int \cos(t) + i\sin(t)\,dt = \int \mathrm{e}^{it}\,dt = i\mathrm{e}^{it}$$

We don't want the whole integral, however, only $\int \cos(t)\,dt$, which is the real part of our answer. To find the real part, expand it first:

$$i\mathrm{e}^{it} = i(\cos(t) + i\sin(t)) = -\sin(t) + i\cos(t)$$

Therefore, the real part is $-\sin(t)$, as expected.

Here's another example, where we complexify the problem by putting the sine or cosine into the complex exponential.

$$\int \mathrm{e}^t \sin(2t)\,dt \quad \Rightarrow \quad \int \mathrm{e}^t(\cos(2t) + i\sin(2t))\,dt = \int \mathrm{e}^{(1+2i)t}\,dt = \frac{1}{1+2i}\mathrm{e}^{(1+2i)t}$$

In this case, we want the imaginary part of the answer only, so we expand the expression:

$$\frac{1}{1+2i}\mathrm{e}^t(\cos(2t) + i\sin(2t)) = \mathrm{e}^t\left(\frac{1}{5} - i\frac{2}{5}\right)(\cos(2t) + i\sin(2t))$$

You might check to see that the imaginary part is given below:

$$\int \mathrm{e}^t \sin(2t)\,dt = -\frac{2}{5}\mathrm{e}^t\cos(2t) + \frac{1}{5}\mathrm{e}^t\sin(2t)$$

Similarly, we can replace the sine or cosine forcing function in a differential equation by using the complex exponential. We might find the algebra easier to navigate. Let's see how that would work.

## Example 1

Find the particular solution to the differential equation:

$$y'' + 2y' + y = \cos(3t)$$

SOLUTION: Rather than solve this problem, we will complexify the right hand side:

$$y'' + 2y' + y = \cos(3t) + i\sin(3t) = \mathrm{e}^{3it}$$

We'll continue as if this is a regular exponential, and use Method of Undetermined Coefficients:

$$y_p = A\mathrm{e}^{3it} \qquad y_p' = 3iA\mathrm{e}^{3it} \qquad y_p'' = -9A\mathrm{e}^{3it}.$$

Substituting into the DE, we get:

$$A\mathrm{e}^{3it}(-9 + 2(3i) + 1) = \mathrm{e}^{3it} \quad \Rightarrow \quad A = \frac{1}{-8+6i}$$

The ansatz was $y_p = A\mathrm{e}^{3it}$, so let's actually expand that to see what it is:

$$\frac{1}{-8+6i}\mathrm{e}^{3it} = \frac{1}{-8+6i}\left(\cos(3t) + i\sin(3t)\right) = \frac{-4-3i}{50}\left(\cos(3t) + i\sin(3t)\right) =$$

$$\left(-\frac{4}{50}\cos(3t) + \frac{3}{50}\sin(3t)\right) + i\left(-\frac{3}{50}\cos(3t) - \frac{4}{50}\sin(3t)\right)$$

Note that we did not actually need to compute the entire answer, only the real part because $\cos(3t)$ is the real part of the exponential. Now, the solution is:

$$y_p = \mathrm{Real}(A\mathrm{e}^{3it}) = -\frac{4}{50}\cos(3t) + \frac{3}{50}\sin(3t)$$

Let's take this one step further and write the result as $R\cos(\omega t - \delta)$:

$$R = \sqrt{\frac{(-4)^2 + 3^2}{50^2}} = \sqrt{\frac{25}{50^2}} = \frac{1}{10}, \qquad \tan(\delta) = \frac{3/50}{-4/50} = \frac{6}{-8}$$

We might make the observation that, using the original constant $A$:

$$\frac{1}{|-8+6i|} = \frac{1}{\sqrt{64+36}} = \frac{1}{10} = R, \quad \text{and} \quad \delta = \arg(-8+6i)$$

so that the $R, \delta$ may be computed directly from the constant $A$ rather than multiplying out the solution. If we have a sine on the RHS, then we either need a slightly different formula, or we can multiply $A\mathrm{e}^{\omega it}$ out (we'll do this in Example 3 below).

## Example 2

Write the particular part of the solution as $R\cos(\omega t - \delta)$, if $y'' + y' - 2y = \cos(2t)$.

SOLUTION: We'll rewrite it first:

$$y'' + y' - 2y = \cos(2t) + i\sin(2t) = \mathrm{e}^{2it}$$

The ansatz and its derivatives are computed:

$$y_p = A\mathrm{e}^{2it} \qquad y_p' = 2iA\mathrm{e}^{2it} \qquad y_p'' = -4A\mathrm{e}^{2it}$$

Putting these into the DE and solve for $A$:

$$A\mathrm{e}^{2it}\left(-4 + 2i - 2(1)\right) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-6+2i}$$

Now,

$$R = \frac{1}{|-6+2i|} = \frac{1}{\sqrt{36+4}} = \frac{1}{\sqrt{40}}, \qquad \delta = \mathrm{Tan}^{-1}(2/-6) = \tan^{-1}(1/3) + \pi$$

The particular solution is:

$$y_p(t) = \frac{1}{2\sqrt{10}}\cos\left(2t - (\tan^{-1}(1/3) + \pi)\right)$$

This was a lot faster than taking $y_p = A\cos(2t) + B\sin(2t)$...

## Example 3

Solve: $y'' + 2y' + y = \sin(3t)$

In this case, we would use exactly the same technique as before, but we go after the imaginary part of the solution at the end. That is, we:

- Replace $\sin(3t)$ by $\mathrm{e}^{3it}$
- Use $y_p = A\mathrm{e}^{3it}$.
- Take the imaginary part of $y_p$ as the solution.

And we can see what our solution is:

$$y_p = -\frac{3}{50}\cos(3t) - \frac{4}{50}\sin(3t)$$

## Example 4

Solve: $y'' + 9y = \sin(3t)$.

If we were to do this using sine and cosine, we would have to guess:

$$y_p = t(A\cos(3t) + B\sin(3t))$$

But wouldn't it be a bit easier to solve using the complex exponential? In that case, we solve the following, again noting that the homogeneous part of the solution is

$$y_h = C_1\cos(3t) + C_2\sin(3t)$$

We re-write the ODE as:

$$y'' + 9y = \mathrm{e}^{3it}$$

And take the following as our ansatz (multiplied by $t$). Recall that we'll actually only need the imaginary part of $y_p$:

$$y_p = At\mathrm{e}^{3it} \qquad y_p' = A\mathrm{e}^{3it} + 3iAt\mathrm{e}^{3it} \qquad y_p'' = 6iA\mathrm{e}^{3it} - 9At\mathrm{e}^{3it}$$

Now,

$$y_p'' + 9y_p = 6iA\mathrm{e}^{3it} \quad \Rightarrow \quad 6iA\mathrm{e}^{3it} = \mathrm{e}^{3it}.$$

From this, we see that:

$$A = \frac{1}{6i} = -\frac{1}{6}i$$

As before, we want the imaginary part of $A\mathrm{e}^{3it}$, which in this case will be:

$$y_p = \mathrm{Imag}(At\mathrm{e}^{3it}) = \mathrm{Imag}\left(-\frac{t}{6}\,i(\cos(3t) + i\sin(3t)\right) = \frac{t}{6}\sin(3t)$$

## Shortcut Summary for Cosine Forcing Functions

Suppose we have the DE:

$$y'' + py' + qy = \cos(\omega t)$$

where we assume $p, q$ are real numbers. If we use our shortcut assumption, we can solve for the constant $A$ coming from the Method of Undetermined Coefficients:

$$y_p = A\mathrm{e}^{i\omega t} \quad \Rightarrow \quad A\mathrm{e}^{i\omega t}(-\omega^2 + i\omega p + q) = \mathrm{e}^{i\omega t}$$

which leaves us with:

$$A = \frac{1}{(q - \omega^2) + i\omega p} = \frac{1}{\alpha + i\beta}$$

We will show that, if we write $y_p = R\cos(\omega t - \delta)$, then

$$R = \frac{1}{|\alpha + i\beta|} \qquad \text{and} \qquad \delta = \arg(\alpha + \beta i) = \arctan\left(\frac{\beta}{\alpha}\right)$$

**Proof:** Compute the real part:

$$\text{Real}\left(\frac{\alpha - \beta i}{\alpha^2 + \beta^2}(\cos(\omega t) + i\sin(\omega t))\right) = \frac{\alpha}{\alpha^2 + \beta^2}\cos(\omega t) + \frac{\beta}{\alpha^2 + \beta^2}\sin(\omega t) = R\cos(\omega t - \delta)$$

with

$$R = \sqrt{\frac{\alpha^2}{(\alpha^2 + \beta^2)^2} + \frac{\beta^2}{(\alpha^2 + \beta^2)^2}} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} = \frac{1}{|\alpha + \beta i|}$$

And for the phase angle $\delta$, the sum of squares terms cancel out leaving us with the angle for $\alpha + \beta i$, or

$$\delta = \arctan\left(\frac{\beta}{\alpha}\right)$$

## Practice:

1. Use the complex exponential to integrate the following:
   (a) $\displaystyle\int \mathrm{e}^{-2t}\cos(t)\,dt$
   (b) $\displaystyle\int \mathrm{e}^{t/2}\sin(3t)\,dt$
2. Use the complex exponential to find $y_p$, given:
   (a) $y'' + 7y = 3\cos(3t)$
   (b) $y'' + y' + 3y = 2\sin(2t)$
   (c) $y'' + 2y' + y = \cos(2t)$
3. Use the complex exponential to find the amplitude and phase angle for the forced response:

$$y'' + 2y' + 2y = \cos(t)$$

## Solutions to the Practice

1. Use the complex exponential to integrate the following:

   (a) $\int e^{-2t}\cos(t)\,dt$
   SOLUTION: We'll take the real part of:

   $$\int e^{-2t}e^{it}\,dt = \int e^{(-2+i)t}\,dt = \frac{1}{-2+i}e^{(-2+i)t}$$

   Expanding this,

   $$\text{Real}\left(e^{-2t}\left(\frac{-2-i}{5}(\cos(t)+i\sin(t))\right)\right) = e^{-2t}\left(-\frac{2}{5}\cos(t)+\frac{1}{5}\sin(2t)\right)+C$$

   (b) $\int e^{t/2}\sin(3t)\,dt$
   SOLUTION: Same idea as before:

   $$\int e^{t/2}e^{3it}\,dt = \int e^{\left(\frac{1}{2}+3i\right)t}\,dt = \frac{1}{\frac{1}{2}+3i}e^{\left(\frac{1}{2}+3i\right)t} = \frac{2}{1+6i}e^{t/2}(\cos(3t)+i\sin(3t))$$

   A little more simplification:

   $$2e^{t/2}\left(\frac{-1-6i}{37}(\cos(3t)+i\sin(3t)\right)$$

   And we take the imaginary part, so that:

   $$\int e^{t/2}\sin(3t)\,dt = e^{t/2}\left(-\frac{12}{37}\cos(3t)+\frac{2}{37}\sin(3t)\right)+C$$

2. Use the complex exponential to find $y_p$, given:

   (a) $y''+7y=3\cos(3t)$
   SOLUTION: The ansatz for $y''+7y=3e^{3it}$ would be $y_p=Ae^{3it}$ (and we'll keep the real part). Substitute into the DE and factor out $Ae^{3it}$:

   $$Ae^{3it}(-9+7)=3e^{3it} \quad \Rightarrow \quad A=\frac{-3}{2}$$

   The particular part of the solution is $y_p=-\frac{3}{2}\cos(3t)$.

   (b) $y''+y'+3y=2\sin(2t)$
   SOLUTION: The ansatz for $y''+y'+3y=2\sin(2t)$ is $y_p=Ae^{2it}$ (imaginary part). Substitute:

   $$Ae^{2it}(-4+2i+3)=2e^{2it} \quad \Rightarrow \quad A=\frac{2}{-1+2i}$$

   We take the imaginary part of the expression below (the complex number has been rationalized):

   $$\frac{-2-4i}{5}(\cos(2t)+i\sin(2t)) \quad \Rightarrow \quad y_p=-\frac{2}{5}\cos(2t)-\frac{2}{5}\sin(2t)$$

   (c) Turned in.

3. Turned in.