Some Trigonometry and Complexification

The following is a little more explanation of the trig that comes up in Chapter 4.

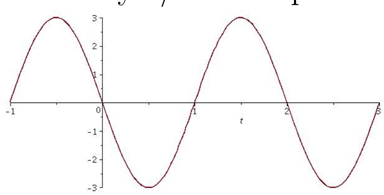
Amplitude, Period, Phase Shift

Given the function $A\cos(\omega t-\delta)$, the amplitude of the function is A, the period is $2\pi/\omega$, the frequency is $\omega/(2\pi)$, the circular frequency is ω, and the phase shift is δ/ω.

So, for example, if we were to graph the function

$$3\cos\left(\pi t+\frac{\pi}{2}\right),$$

the amplitude is 3, the period is 2, and the phase shift is $-\frac{1}{2}$, which would translate as "shift to the left by 1/4 of the period". The result:



Cosine Sum Formula

You might recall the cosine sum formula:

$$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$$

Which in fact is the real part of the complex number:

$$\cos(x-y)+i\sin(x-y)=\mathrm{e}^{i(x-y)}=\mathrm{e}^{ix}\mathrm{e}^{-iy}=(\cos(x)+i\sin(x))(\cos(-y)+i\sin(-y))$$

Continuing the RHS, we get:

$$\mathrm{e}^{i(x-y)}=(\cos(x)\cos(y)+\sin(x)\sin(y))+i(\sin(x)\cos(y)-\sin(y)\cos(x))$$

The real part gives us the cosine difference formula, and the imaginary part gives us the sine difference formula.

Writing a Trig Sum

Using this, we see that:

$$R\cos(\omega t-\delta)=(R\cos(\delta))\cos(\omega t)+(R\sin(\delta))\sin(\omega t)$$

This implies that we can write the following sum as a single cosine function:

$$A\cos(\omega t)+B\sin(\omega t)=R\cos(\omega t-\delta)\quad\Leftrightarrow\quad A=R\cos(\delta)\qquad B=R\sin(\delta)$$

or, given A, B then:

$$R = \sqrt{A^2 + B^2} \quad \text{and} \quad \delta = \mathrm{Tan}^{-1}\left(\frac{B}{A}\right)$$

(the inverse tangent is the four quadrant version). As a shortcut to remembering this, let

$$z = A + iB$$

then R, δ come from the magnitude and argument of z.

Side Remark: Careful with the parentheses...

If we write the periodic function as $R\cos(\omega(t-\delta))$, then δ itself is the phase shift, but then we don't have the form for the cosine sum formula.

Worked Examples

1. Rewrite the expression as $R\cos(\omega t - \delta)$:

   $$\cos(\sqrt{3}\,t) + 2\sin(\sqrt{3}\,t)$$

   SOLUTION: Thinking of the complex number $z = 1 + 2i$, then $R = \sqrt{1^2 + 2^2} = \sqrt{5}$, and

   $$\delta = \tan^{-1}(2) \approx 1.1 \text{ rad}$$

   Therefore,

   $$\cos(\sqrt{3}\,t) + 2\sin(\sqrt{3}\,t) = \sqrt{5}\cos(\sqrt{3}\,t - 1.1)$$

   Now it is easy to analyze the sum as a single periodic function- It has an amplitude of $\sqrt{5}$, a period of $2\pi/\sqrt{3}$, and a phase shift of $1.1/\sqrt{3}$ radians.

2. Rewrite the expression as a single cosine:

   $$-\cos(3t) + \sin(3t)$$

   SOLUTION: Thinking of $z = -1 + i$, the magnitude is $\sqrt{2}$ and the inverse tangent gives $-\pi/4$. BUT, we should notice that our point is actually in the second quadrant, so the argument is actually $\pi - \pi/4 = 3\pi/4$ (we should use the "four quadrant" inverse tangent).

   Therefore, the sum is equal to: $\sqrt{2}\cos\left(3t - \frac{3\pi}{4}\right)$.

   The amplitude is $\sqrt{2}$, the period is $2\pi/3$, and the phase shift is $\pi/4$.

Exercises

1. Write the expression $3\cos(5t) - 2\sin(5t)$ in the form $R\cos(\omega t - \delta)$.
2. Write the expression $-\cos(2t) + \sin(2t)$ in the form $R\cos(\omega t - \delta)$.
3. Write the expression $-\cos(t) - 3\sin(t)$ in the form $R\cos(\omega t - \delta)$.