Final Exam Review: Math 244

For the Spring 2021 semester, the exam will be available on Canvas from midnight Saturday, May 15 to Tuesday, May 17 at 3:30AM (theoretically, you could start the exam at 11:59PM Monday May 17).

- **What do I need?**

  You might keep a table of the Laplace transforms handy, as well as a basic scientific calculator. As you study, keep notes of the things you're not sure about so you don't have to search through your textbook if you get stuck.

- **Format**

  The format will be identical to our last exam; the exam page in Canvas is just there to sign your name, download the file, then upload your solutions.

- **Weight of Topics**

  There is some crossover between the latest material and Chapters 2 and 3, but generally speaking the exam will be weighted about 30% on the latest material (topics from Ch 7), about 30% on Chapters 5 and 6, and about 40% on Chapters 2 and 3.

- **What should I study?**

  Study these notes, but also be sure to go back over your old exams and be sure that you're able to answer all the questions- They may show up again! Once you've gone through the old exams, page through the old study guides. For the new material, I've included a separate summary sheet.

- **Points**

  I want to see your work! It is your logical chain of arguments and algebra/calculus that is the most important thing for you to show. If you just write down the answer with no supporting work, or if your work is completely unorganized, you will lose points. However, if you have the wrong answer but have correct reasoning, you can get partial credit.

- **Timing**

  As for length, the exam is slightly longer than a normal exam- I think of it as an exam and a half. For time, a couple of hours is normally plenty, but there will be a time limit of 3 hours (plus 10 minutes for you to scan/upload it). **Everyone should upload their solution to Canvas in the time allotted**.

  I think time management can be an issue, so you might set some preliminary timers for yourself if you've had trouble with that in the past- Maybe a timer at one hour, at two hours, and at 2.5 hours to remind you to start wrapping up. If you do have any issues with uploading your exam, you can always send me a copy.

- Keep a copy of your scan somewhere safe (don't delete it)- it has a record of when it was created.

Review Questions

1. Solve (use any method if not otherwise specified):

   (a) $-t\cos(t)\,dt + (2x - 3x^2)\,dx = 0$

   (b) $y'' + 2y' + y = \sin(3t)$

   (c) $y' = y(y-1)$

   (d) $y'' - 3y' + 2y = e^{2t}$

   (e) $y' = \sqrt{t}e^{-t} - y$

   (f) $x' = 2 + 2t^2 + x + t^2 x$

   (g) $\begin{aligned} x_1' &= 2x_1 + 3x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$

   (h) $(y\cos(x) + 2xe^y) + (\sin(x) + x^2e^y - 1)y' = 0$

2. Use the ansatz $y = t^r$ to get the general solution to the linear DE: $t^2y'' - 2ty' - 10y = 0$

3. Show that using $v = y/x$, the following equation becomes separable as a DE in v. NOTE: You do not need to solve the differential equation.

$$\frac{dy}{dx} = \frac{3x - 4y}{y - 2x}$$

4. Show that with the substitution $w = y^3$, the following equation becomes linear in w. NOTE: You do not need to solve the differential equation.

$$\frac{dy}{dx} + 3xy = \frac{x}{y^2}$$

5. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:

$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

6. If $y' = y(1-y)(2-y)(3-y)(4-y)$ and $y(0) = 5/2$, determine what y does as $t \to \infty$. Hint: Do not try to actually solve the DE.

7. If y_1, y_2 are a fundamental set of solutions to $t^2y'' - 2y' + (3+t)y = 0$ and if $W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(4)$.

8. Given $t^2y'' - 2y' + (3+t)y = 0$, with $y(-1) = -1$ and $y'(-1) = 2$, on what interval can we guarantee that a unique solution exists? (Be explicit about what you're computing and why).

9. Let $y'' - 6y' + 9y = F(t)$. For each $F(t)$ listed, give the *form* of the general solution using undet. coeffs (do not solve for the coefficients).

   (a) $F(t) = 2t^2$

   (b) $F(t) = te^{-3t}\sin(2t)$

   (c) $F(t) = t\sin(2t) + \cos(2t)$

   (d) $F(t) = 2t^2 + 12e^{3t}$

10. Give Newton's law of cooling in words, then as a differential equation, then solve it!

11. A spring is stretched 0.1 m by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 0.05 m below its resting equilibrium and released with a downward velocity of 0.1 m/s, determine its position u at time t.

12. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0$, $x_0 = 1$. Find the recurrence relation and write the first 5 terms of the expansion of y.

13. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0$, $x_0 = 1$ (the same as the previous DE), with $y(1) = 1$ and $y'(1) = 2$. Use the method of derivatives to compute the first 5 terms of the Taylor series.

14. Use the *definition* of the Laplace transform to determine $\mathcal{L}(f)$: $f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6-t, & 2 < t \end{cases}$.

15. Determine the Laplace transform:

(a) $t^2 e^{-9t}$ (b) $u_5(t)(t-2)^2$ (c) $e^{3t}\sin(4t)$ (d) $e^t\delta(t-3)$

16. Find the inverse Laplace transform:

    (a) $\dfrac{2s-1}{s^2-4s+6}$ (b) $\dfrac{7}{(s+3)^3}$ (c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$ (d) $\dfrac{3s-2}{(s-4)^2-3}$

17. Solve the given initial value problems using Laplace transforms:

    (a) $y'' + 2y' + 2y = 4t$, $y(0) = 0$, $y'(0) = -1$
    (b) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
    (c) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$ (You may write the solution as a convolution)

18. Consider $t^2 y'' - 4ty' + 6y = 0$. Using $y_1 = t^2$ as one solution, find y_2 by computing the Wronskian two ways.

19. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

    (a) $y' - \frac{1}{2}y = e^{2t}$ $y(0) = 1$
    (b) $y' = \frac{1}{2}y(3-y)$
    (c) $y'' + 2y' + y = 0$, $y(0) = \alpha, y'(0) = 1$
    (d) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$
    (e) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1$.

20. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down.

21. True or False (and explain): Every separable equation is also exact. If true, is one way easier to solve over the other?

22. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

23. Rewrite the following differential equations as an equivalent system of first order equations. If it is an IVP, also determine initial conditions for the system.

    (a) $y'' - 3y' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$.
    (b) $y''' - 2y'' - y' + 4y = 0$
    (c) $y'' - yy' + t^2 = 0$

24. Convert one of the variables in the following systems to an equivalent higher order differential equation, and solve it (be sure to solve for both x and y): $\begin{aligned} x' &= 4x + y \\ y' &= -2x + y \end{aligned}$

25. Solve the previous system by using eigenvalues and eigenvectors.

26. Verify by direct substitution that the given power series is a solution of the differential equation:

$$y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad (x+1)y'' + y' = 0$$

27. Convert the given expression into a single power series:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=2}^{\infty} n a_n x^{n-2} + 3\sum_{n=1}^{\infty} a_n x^n$$

28. Find the recurrence relation for the coefficients of the series solution to $y'' - (1+x)y = 0$ at $x_0 = 0$.

29. Find the first 5 non-zero terms of the series solution to $y'' - (1+x)y = 0$ if $y(0) = 1$ and $y'(0) = -1$ (use derivatives).

30. Let $y'' + \omega^2 y = \cos(\alpha t)$.

    (a) What values of ω, α will result in *beating*? Write the homogenous part of the solution, then give the *form* of the particular part of the solution from Method of Undetermined Coefficients.

    (b) Repeat the first part, except for *resonance*.

31. Let $y'' + \alpha y' + y = 0$. Find (all) values of α for which the solution is *underdamped*, *overdamped*, and *critically damped*.

32. Let $y'' + y' + y = \cos(2t)$.

    (a) If we complexify the problem, how is the right side of the equation changed? How is the ansatz changed?

    (b) Using your previous answer, find the amplitude and the phase shift of the forced response, $y_p(t) = R\cos(\omega t - \delta)$.

33. Given $y'' + \alpha y' + y = \cos(2t)$, find α that will maximize the amplitude of the forced response. (You might do the previous problem first!)

34. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

35. Solve the linear system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, if A is as defined below:

    (a) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$

36. For each system $\mathbf{x}' = A\mathbf{x}$, the matrix A depends on the parameter α. Find how the classification of the origin changes depending on α, indicating where the solution lies on the Poincare Diagram.

    (a) $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$ (b) $\begin{bmatrix} \alpha & 2 \\ 3 & 1 \end{bmatrix}$

37. We have two tanks, A and B with 20 and 30 gallons of fluid, respectively. Water is being pumped into Tank A at a rate of 2 gallons per minute, 2 ounces of salt per gallon. The well-mixed solution is pumped out of Tank A and into Tank B at a rate of 4 gallons per minute. Solution from Tank B is entering Tank A at a rate of 2 gallons per minute. Water is being pumped into Tank B at k gallons per minute with 3 ounces of salt per gallon. The solution is being pumped out of tank B at a total rate of 5 gallons per minute (2 of them are going into tank A).

    - What should k be in order for the amount of solution in Tank B to remain at 30? Use this value for the remaining problems.
    - Write the system of differential equations for the amount of salt in Tanks A, B at time t. Do not solve.
    - Find the equilibrium solution and classify it.

38. For the following system, (i) Sketch the nullclines (with directions along the nullclines), (ii) determine the equilibria, (iii) perform an equilibrium point analysis for each equilibrium solution.

    $$\begin{aligned} x' &= 1 + 2y \\ y' &= 1 - 3x^2 \end{aligned}$$

39. For the following system, we'll only analyze the **first quadrant**. In that quadrant, (i) Sketch the nullclines (with directions along the nullclines), (ii) determine the equilibria, (iii) perform an equilibrium point analysis for each equilibrium solution.

$$\begin{aligned} x' &= x(2 - x - y) \\ y' &= y(y - x^2) \end{aligned}$$

40. For the following system, (i) Sketch the nullclines (with directions along the nullclines), (ii) determine the equilibria, (iii) perform an equilibrium point analysis for each equilibrium solution.

$$\begin{aligned} x' &= 2 - x - y \\ y' &= y - |x| \end{aligned}$$

41. For the following DE, just find the equilibria and perform the equilbrium point analysis.

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$$