Last time:

Solve $ay'' + by' + cy = 0$.

- $b^2 - 4ac > 0$

$$y(t) = C_1 \mathrm{e}^{r_1 t} + C_2 \mathrm{e}^{r_2 t}$$

- $b^2 - 4ac < 0$, with $r = \alpha + \beta i$

$$y(t) = C_1 \mathrm{Re}(\mathrm{e}^{rt}) + C_2 \mathrm{Im}(\mathrm{e}^{rt}) = \mathrm{e}^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

- $b^2 - 4ac = 0$

$$y(t) = C_1 \mathrm{e}^{rt} + C_2 t \mathrm{e}^{rt} = \mathrm{e}^{rt}(C_1 + C_2 t)$$

Today: How to solve $ay'' + by' + cy = f(t)$

Notation

The solution to

$$ay'' + by' + cy = 0 \text{ or } L(y) = 0 \quad \Rightarrow \quad y_h(t)$$

The solution to

$$ay'' + by' + cy = g(t) \text{ or } L(y) = g(t) \quad \Rightarrow \quad y_p(t)$$

Some Theory

- To solve $L(y) = g(t)$, the full general solution will be $y(t) = y_h(t) + y_p(t)$.

$$L(y_h(t) + y_p(t)) = L(y_h) + L(y_p) = 0 + g(t) = g(t)$$

- Constants (to solve the IVP) will be in y_h:

$$L(cy_h) = cL(y_h) = 0 \qquad L(cy_p) = cL(y_p) = cg(t) \neq g(t)$$

A bit more theory

We solve:

$$L(y) = g_1(t) + g_2(t) + g_3(t) + \cdots g_n(t)$$

by breaking it up into n different smaller problems:

$$L(y) = 0, \quad L(y) = g_1(t) \quad L(y) = g_2(t) \quad \cdots \quad L(y) = g_n(t)$$

So the overall solution will be:

$$y(t) = y_h(t) + y_{p_1}(t) + \cdots + y_{p_n}(t)$$

Example

$$y'' - y = t$$

- $y_h(t)$: $r^2 - 1 = 0 \Rightarrow y_h(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{t}$.
- Observation:
  The derivative(s) of a polynomial is a polynomial.
  Guess $y_p(t) = At + B$, and determine A, B.

$$y_p = At + B, y_p' = A, y_p'' = 0 \quad \Rightarrow \quad 0 - (At + B) = t$$

Therefore, $A = -1$ and $B = 0$.

Overall:

$$y(t) = y_h(t) + y_p(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{t} - t$$

Nice $f(t)$

Some classes of the forcing function are particularly nice because the derivative of:

- $P_n(t)$ is another polynomial (deg $n-1$).
- $P_n(t)e^{\alpha t}$ is $p_n(t)e^{\alpha t}$
- $P_n(t)\sin(\beta t)$ or $P_n(t)\cos(\beta t)$ is $p_n(t)\sin(\beta t)$ and $p_n(t)\cos(\beta t)$
- $P_n(t)e^{\alpha t}\sin(\beta t)$ or $P_n(t)e^{\alpha t}\cos(\beta t)$ is $p_n(t)e^{\alpha t}\sin(\beta t)$ and $p_n(t)e^{\alpha t}\cos(\beta t)$

The Method of Undetermined Coefficients

See this for $g(t)$	Guess this for $y_p(t)$
$t^2 + t$	$At^2 + Bt + C$
e^{3t}	$A\mathrm{e}^{3t}$
$\sin(3t)$	$A\cos(3t) + B\sin(3t)$
$t\mathrm{e}^{2t}$	$(At + B)\mathrm{e}^{2t}$
$t\sin(3t)$	$(At + B)\sin(3t) + (Ct + D)\cos(3t)$
$t\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t)$	$\mathrm{e}^{-t}((At + B)\sin(t) + (Ct + D)\cos(t))$

There is one more thing to keep in mind, but we'll wait on that.

Example

Solve $y'' + 3y' + 2y = \mathrm{e}^t + t^2 + 2$.

- $r^2 + 3r + 2 = 0$ gives $y_h(t) = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{-t}$.
- $y'' + 3y' + 2y = \mathrm{e}^t$ by guessing $y_p = A\mathrm{e}^t$:
  $A\mathrm{e}^t + 3A\mathrm{e}^t + 2A\mathrm{e}^t = \mathrm{e}^t \quad \Rightarrow \quad A = \dfrac{1}{6}$
- $y'' + 3y' + 2y = t^2 + 2$ by $y_p = at^2 + bt + c$.
  $2a + 3(2at + b) + 2(at^2 + bt + c) = t^2 + 2$

$$\begin{array}{lll} t^2: & 2a = 1 & \Rightarrow a = 1/2 \\ t: & 6a + 2b = 0 & \Rightarrow b = -3/2 \\ const: & 2a + 3b + 2c = 2 & \Rightarrow c = 11/4 \end{array}$$

$$y(t) = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^t + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{11}{4}$$

Example

Solve $y'' + 3y' + 2y = \mathrm{e}^{-t}$.

SOLUTION: Guess $y_p = A\mathrm{e}^{-t}$, $y' = -A\mathrm{e}^{-t}$, $y_p'' = A\mathrm{e}^{-t}$:

$$A\mathrm{e}^{-t} - 3A\mathrm{e}^{-t} + 2\mathrm{e}^{-t} = \mathrm{e}^{-t}$$

$$0 = \mathrm{e}^{-t}$$

Solve $y'' + 3y' + 2y = \mathrm{e}^{-t}$

To fix this, multiply your guess by t:

$$y_p = At\mathrm{e}^{-t} \qquad y_p' = A\mathrm{e}^{-t} - At\mathrm{e}^{-t} \qquad y_p'' = -2A\mathrm{e}^{-t} + At\mathrm{e}^{-t}$$

The differential equation:

$$\mathrm{e}^{-t}\left[(-2A + At) + 3(A - At) + 2At\right] = \mathrm{e}^{-t} \quad \Rightarrow \quad A = 1$$

$$y(t) = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{-t} + t\mathrm{e}^{-t}$$

The Full Method

To solve $ay'' + by' + cy = g_i(t)$,

If $g_i(t)$ is :	The ansatz for y_{p_i} :
$P_n(t)$	$t^s(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s \mathrm{e}^{\alpha t}(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)\mathrm{e}^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$t^s \mathrm{e}^{\alpha t}[\cos(\beta t)(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0) +$ $\sin(\beta t)(b_n t^n + \ldots + b_2 t^2 + b_1 t + b_0)]$

where s is chosen so that no part of y_{p_i} is part of the homogeneous solution.
Substitute y_{p_i} into the DE and solve for the coefficients.

Example

- $y'' - 3y' - 4y = 3\mathrm{e}^{2t}$ $(r = 4, -1)$
  SOLUTION: $y_p = A\mathrm{e}^{2t}$
- $y'' - 3y' - 4y = \cos(t)$ $(r = 4, -1)$.
  SOLUTION: $y_p = A\cos(t) + B\sin(t)$
- $y'' - 3y' - 4y = 6t\mathrm{e}^{t}\sin(2t)$ $(r = 4, -1)$
  SOLUTION: $y_p = \mathrm{e}^{t}((At + B)\cos(2t) + (Ct + D)\sin(2t))$

Examples

- $y'' - y' - 2y = 3t^2$: $r = -1, 2$,
  SOLUTION: $y_p = At^2 + Bt + C$.
- $y'' - y' - 2y = \mathrm{e}^{2t}$: $r = -1, 2$
  SOLUTION: $y_p = (A\mathrm{e}^{2t})t = At\mathrm{e}^{2t}$
- $y'' - y' - 2y = t\mathrm{e}^{-t}$: $r = -1, 2$.
  SOLUTION: $y_p = t(At + B)\mathrm{e}^{-t} = (At^2 + Bt)\mathrm{e}^{-t}$.

Examples

- $y'' - 4y' + 4y = t\mathrm{e}^{2t}$: $r = 2, 2$.
  SOLUTION: Initial $y_p = (At + B)\mathrm{e}^{2t}$.
  Multiply by t: $y_p = t(At + B)\mathrm{e}^{2t}$.
  Multiply by t: $y_p = t^2(At + B)\mathrm{e}^{2t}$
- $y'' - y' = 3t + 5$: $r = 0, 1$
  SOLUTION: Initial $y_p = (At + B)$
  Multiply by t: $y_p = t(At + B)$.
- $y'' + 2y' + y = t\mathrm{e}^{t}\sin(2t)$. $r = -1, -1$.
  SOLUTION:

$$y_p = \mathrm{e}^{t}\left[(At + B)\sin(2t) + (Ct + D)\cos(2t)\right]$$

Examples

- $y'' + 4y = t^2 \sin(2t) + (6t+7)\cos(2t)$ $(r = \pm 2i)$

  $$y_p = (At^2 + Bt + C)\sin(2t) + (Dt^2 + Et + F)\cos(2t)$$

  Multiply by t:
  $y_p = t(At^2 + Bt + C)\sin(2t) + t(Dt^2 + Et + F)\cos(2t)$
- $y'' + 2y' + 5y = 3t\mathrm{e}^{-t}\cos(2t)$ $(r = -1 \pm 2i)$

  $$y_p = \mathrm{e}^{-t}(At + B)\cos(2t) + \mathrm{e}^{-t}(Ct + D)\sin(2t)$$

  But, we need to multiply through by t.

  $$y_p = t\mathrm{e}^{-t}\left[(At + B)\cos(2t) + (Ct + D)\sin(2t)\right]$$