General nonlinear systems: Two techniques

For a general nonlinear system of first order differential equations, there is no general way to obtain an analytic solution.

Do you feel lucky?

It may be possible to solve the nonlinear system of differential equations by constructing dy/dx:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x,y)}{f(x,y)}$$

but these would be very special DEs. Typically, we would have to resort to numerically approximating solutions, and going for more qualitative methods: (1) Equilibrium Point Analysis, and (2) Method of Nullclines.

Equilibrium Point Analysis

Given a system of nonlinear autonomous DEs:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad \text{or} \qquad \mathbf{x}'(t) = F(\mathbf{x})$$

find the equilibrium solutions (call them (a,b)). We then linearize the nonlinear system at each equilibrium. That is,

$$\begin{aligned} x' &\approx \; f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ y' &\approx \; g(a,b) + g_x(a,b)(x-a) + g_y(a,b)(y-b) \end{aligned}$$

At equilibrium, $f(a,b) = g(a,b) = 0$, and if we make the local change of coordinates $u = x - a$, $v = y - b$, then the **linearized system at** (a,b):

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

We can then use the Poincaré Diagram to determine the local behavior. We must use some caution in the case of centers and degenerate nodes, however. Because the linearization is an *approximation* of the true solution, the actual solutions are of a slightly perturbed system. This means that while the linearization gives a center, the true solution may be a center or a spiral (we would use a computer simulation to see what we actually get).

Example:

Consider the following system:

$$\begin{aligned} x' &= \; x - 0.5xy & &= x(1-0.5y) \\ y' &= \; -0.75y + 0.25xy & &= y(-0.75 + 0.25x) \end{aligned}$$

Analyze the behavior of the solutions of the system by using local linearization.

SOLUTION: Equilibria first. Remember that BOTH equations must be zero:

$$\begin{array}{rl} x(1-0.5y) & = 0 \\ y(-0.75+0.25x) & = 0 \end{array} \quad \Rightarrow \quad \text{From Eq 1: } x = 0 \text{ or } y = 2 \quad \Rightarrow \quad (0,0) \text{ or } (3,2)$$

We have only two equilibria for this system, $(0,0)$ and $(3,2)$. Now compute the "Jacobian matrix" of partial derivatives:

$$\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1-0.5y & -0.5x \\ 0.25y & -0.75+0.25x \end{bmatrix}$$

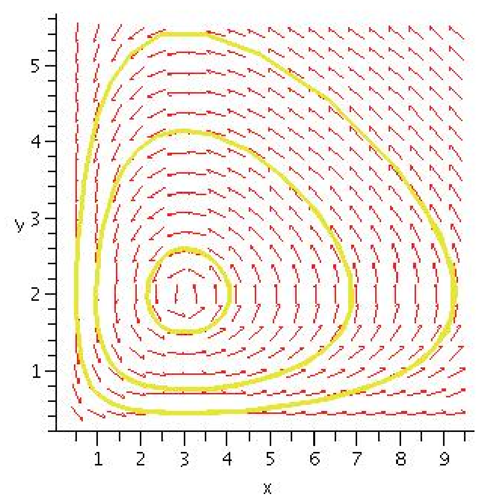
Linearizing about the two equilibrium gives (in order):

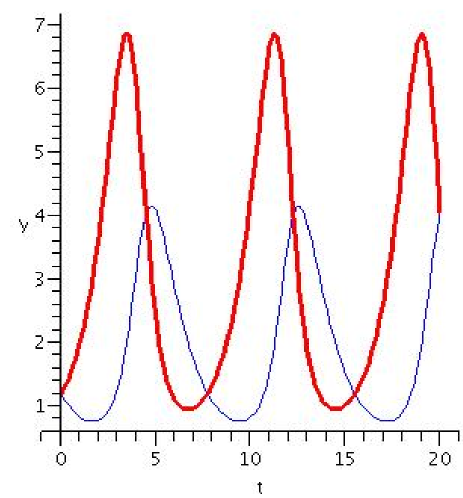
$$\begin{bmatrix} 1 & 0 \\ 0 & -0.75 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1.5 \\ 0.5 & 0 \end{bmatrix}$$

In the first case, the trace is $1/4$ and the determinant is $-3/4$. By the Poincare diagram, the origin is a SADDLE.

At the point $(3,2)$, the trace is 0 and the determinant is positive: We have a CENTER. We should check the direction field to verify our analysis.

In the figure below, we first show the (x,y) plane, then in the next figure, we plot $x(t)$ and $y(t)$ versus t.





Method of Nullclines

If we have the system given to the left, then the two curves (independently) are the two nullclines. We note that the intersection of the curves ($f(x,y) = 0$ with $g(x,y) = 0$) gives the set of equilibrium solutions.

$$\begin{array}{rl} x' & = f(x,y) \\ y' & = g(x,y) \end{array} \qquad \text{or} \qquad \begin{array}{l} f(x,y) = 0 \\ g(x,y) = 0 \end{array}$$

To do this method, along the points where $f(x,y) = 0$, the direction of motion for the solution curves is up or down. Similarly, along points where $g(x,y) = 0$, the direction of motion is right or left. We've done several examples in the videos of how to determine these, and in the most recent videos, we go through the analysis for the example above.