M&M Death and Immigration[1]

Today we're using M&M candies as a model of a population. The candies almost always have a white "m" imprinted on one side and not the other.

1 Model 1: Death Model

Gently shake the container of 50 M&Ms out onto a flat surface like a desk (or you might want to use a paper plate to catch the M&Ms and keep them clean as well). We determine for each M&M if it lives or dies. If the m shows on top this M&M dies, otherwise there is life for this M&M. Upon death you should remove the M&M from the population (set these aside as we will need them for another experiment), count and note down the number of M&Ms who survive in Table 1, and thus put fewer M&Ms back into your container for the next iteration. Do this over and over and record your results.

Questions:

1. Offer up a description (in words) of what should happen. Care to make a prediction about what happens in the long run?
2. Now perform the experiment. In the table, record what happened.
3. Compare your description/prediction with what actually happened.

Death of M&Ms

Iteration	# M&Ms	Sample
0	50	50
1		25
2		12
3		9
4		2
5		0
6		
7		
8		

Attempt at a mathematical model: Now attempt to build a mathematical model of this situation. Let's define the number of M&Ms alive at the start of iteration n as $a(n)$. What values would n take? What would $a(0)$ be in this case? Do you "know" what $a(1)$ would be?
Questions:

1. Based on the observations and your assumptions, produce a reasonable formula for $a(n)$. That is, offer up a discrete function $a(n)$, in the one variable, n, for $n = 0, 1, 2, \ldots$.
2. How will you measure your "success" as a modeler in this situation?

[1]Adapted from 1-1-M&M-DeathImmigrationParameterEstimation, SIMIODE modeling scenario by Dr. Brian Winkel, Professor Emeritus, USMA, Director SIMIODE, and Dr. Karen Bliss' re-write.

2 Model 2: Death with Immigration

This time we will permit immigration to recover the damage to the M&M population from death. Again, start with 50 M&Ms, and as before Gently shake the M&Ms out onto the flat surface to determine for each M&M if it lives or dies. Remember, if the m shows on top this M&M dies; otherwise, there is life for this M&M. Upon death you should remove the M&M from the population (set these aside).

VERY IMPORTANT CHANGE: In addition to putting the M&Ms which survived back into the cup, add 10 immigrants AT EACH ITERATION. You can use those from your "death pile." Do this over and over and record your results.

Death with Immigration

Iteration	# M&Ms	Sample
0	50	50
1		35
2		33
3		27
4		24
5		24
6		21
7		20
8		21
9		20
10		23

Attempt at a mathematical model: Now attempt to build a mathematical model of this situations. Let's define the number of M&Ms alive at the start of iteration n as $b(n)$.

Produce a reasonable formula for $b(n)$ (we'll model $b(n+1)$ in the next step. That is, offer up a discrete function $b(n)$, in the one variable, n, for $n = 0, 1, 2, \ldots$.

$$b(n) = \underline{\hspace{6cm}} .$$

3 Another Approach: Modeling Change

It is hard to come up with a model formula for the death and immigration model directly- Maybe we need to think about things a little differently. Let's focus on how we get from one generation to the next.

We seem pretty confident that in each iteration we lose about half of our M&Ms (again–why is that so?), but then but we then add 10 M&Ms as immigrants. You should be able to tell what the $(n+1)$th generation, $b(n+1)$, looks like in terms of the nth generation, $b(n)$, and immigration. Here's what we want from our model: "The value of $b(n+1)$ is to be half of $b(n)$ due to death plus 10 due to immigration." Fill in the following:

$$b(n+1) = \underline{\hspace{10cm}} .$$

4 Another Approach: Continuous Dynamics

Let us examine our first model below:

$$a(n+1) = \frac{1}{2}a(n)$$

If we subtract $a(n)$ from both sides, we can model only the change from step n to step $n+1$:

$$a(n+1) - a(n) = \frac{1}{2}a(n) - a(n) = -\frac{1}{2}a(n)$$

Suppose instead of generational increments of 1, i.e. going from n to $n+1$, we were to go in time increments of Δn, e.g., $n = 0.01$. In this case we would not lose half of our population in a time interval of 1. Rather we would lose a fraction, $(\Delta n) \cdot 0.5$, of our population in a small time interval of length Δn.

$$a(n + \Delta n) - a(n) = -(\Delta n) \cdot \frac{1}{2} \cdot a(n) \tag{1}$$

With the introduction of Δn we sense a nearness of the calculus muse, for we can take our approximate population changes over smaller and smaller time intervals of length Δn. Indeed, in (1) we could divide both sides by Δn :

$$\frac{a(n + \Delta n) - a(n)}{\Delta n} = -\frac{1}{2}a(n). \tag{2}$$

and in (2) take the limit as $\Delta n \to 0$, i.e. we take smaller and smaller increments of time over which M&Ms die and immigration takes place:

$$\lim_{\Delta n \to 0} \frac{a(n + \Delta n) - a(n)}{\Delta n} = \lim_{\Delta n \to 0} -\frac{1}{2}a(n).$$

Note that the left hand side is just the definition of a derivative. The expression inside the limit on the right hand side is constant with respect to Δn, so we have:

$$\frac{da}{dn} = -\frac{1}{2} \cdot a(n) \tag{3}$$

With the initial condition and converting fully over to continuous time, we have:

$$\frac{da}{dt} = -\frac{1}{2}a(t), \qquad a(0) = 50. \quad \Rightarrow \quad a(t) = 50\mathrm{e}^{-t/2} \tag{4}$$

Now apply this approach to your $b(n+1)$.