

# Solutions to the Review Questions

## Short Answer/True or False

1. True or False, and explain:

(a) If  $y' = y + 2t$ , then  $0 = y + 2t$  is an equilibrium solution.

False.  $y(t)$  must be constant.

(b) Let  $\frac{dy}{dt} = 1 + y^2$ . The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all  $t$ . (If true, say why. If False, solve the DE).

False. The E & U Theorem tells that a unique solution will exist for any initial condition, but it does not say on what interval the solution will exist.

(c) If  $y' = \cos(y)$ , then the solutions are periodic.

FALSE. The solution to an autonomous equation can do what?

(d) All autonomous equations are separable.

True. Any autonomous equation can be written as  $y' = f(y) \cdot 1$ .

(e) All separable equations are exact.

True. If the equation is separable, then  $y' = f(y)g(x)$ , which can be written:

$$\frac{1}{f(y)} \frac{dy}{dx} = g(x) \quad \Rightarrow \quad -g(x) + \frac{1}{f(y)} \frac{dy}{dx} = 0$$

(I'll let you check that that is exact).

2. The Existence and Uniqueness Theorems:

• Linear:  $y' + p(t)y = g(t)$  at  $(t_0, y_0)$ :

SOLUTION: See the text.

• General Case:  $y' = f(t, y)$ ,  $(t_0, y_0)$ :

SOLUTION: See the text.

3. To solve  $y' = y^{1/3}$ , we separate variables:

$$y^{-1/3} dy = dt$$

Before going further, it is good practice to note that the previous step is valid, *as long as*  $y \neq 0$ . The case that  $y = 0$  can be taken separately- In fact, we see that  $y(t) = 0$  is an equilibrium solution that satisfies the initial condition.

Going on, we integrate:

$$\frac{3}{2}y^{2/3} = t + C_1 \quad \Rightarrow \quad y^{2/3} = \frac{2}{3}t + C_2 \quad \Rightarrow \quad y = \left(\frac{2}{3}t + C_2\right)^{3/2}$$

We can solve for  $C_2$  using the initial condition.

Of course, the Existence and Uniqueness Theorem cannot be “violated” since it is a theorem, but in this case, the *hypotheses* are not satisfied:

$$y' = f(t, y) \quad \Rightarrow \quad f(t, y) = y^{1/3}$$

In this case,  $f$  is continuous at  $(0, 0)$  but  $\partial f/\partial y$  is not.

**Solve:**

1.  $\frac{dy}{dx} = \frac{x^2 - 2y}{x}$

SOLUTION:  $y = \frac{1}{4}x^2 + \frac{C}{x^2}$

2.  $(x + y) dx - (x - y) dy = 0$ . Hint: Let  $v = y/x$ .

Given the hint, rewrite the DE:

$$\frac{dy}{dx} = \frac{x + y}{x - y} = \frac{1 + (y/x)}{1 - (y/x)} = \frac{1 + v}{1 - v}$$

With the substitution  $xv = y$ , we get the substitution for  $dy/dx$ :

$$v + xv' = y'$$

So that the DE becomes:

$$v + xv' = \frac{1 + v}{1 - v} \Rightarrow xv' = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v} = \frac{1 + v^2}{1 - v}$$

The equation is now separable:

$$\frac{1 - v}{1 + v^2} dv = \frac{1}{x} dx \Rightarrow \int \frac{1}{1 + v^2} dv - \int \frac{v}{1 + v^2} dv = \ln|x| + C$$

Therefore,

$$\tan^{-1}(v) - \frac{1}{2} \ln(1 + v^2) = \ln|x| + C$$

Lastly, back-substitute  $v = y/x$ .

3.  $\frac{dy}{dx} = \frac{2x + y}{3 + 3y^2 - x}$   $y(0) = 0$ .

SOLUTION:

$$-x^2 - xy + 3y + y^3 = 0$$

4.  $\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}$

SOLUTION:  $x^2y + xy^2 + x = c$

5.  $\frac{dy}{dt} = 2 \cos(3t)$   $y(0) = 2$

SOLUTION:  $y(t) = \frac{2}{3} \sin(3t) + 2$ , and the solution is valid for all time.

6.  $y' - \frac{1}{2}y = 0$   $y(0) = 200$ . State the interval on which the solution is valid.

This is linear and separable. As a linear equation, the solution will be valid on all  $t$  (since  $p(t) = -\frac{1}{2}$ ).

The solution is  $y(t) = 200e^{(1/2)t}$

7. This is separable (or Bernoulli):

$$\int y^{-2} dy = \int (1 - 2x) dx \Rightarrow -\frac{1}{y} = x - x^2 + C$$

Put in the initial condition (IC):  $6 = 0 + C$ . Now finish solving explicitly:

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x - 3)(x + 2)}$$

The solution is valid on the interval  $(-2, 3)$ .

8.  $y' - \frac{1}{2}y = e^{2t}$       $y(0) = 1$

SOLUTION:  $y(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{(1/2)t}$

9.  $y' = \frac{1}{2}y(3 - y)$

SOLUTION:  $y(t) = \frac{3}{(1/A)e^{-(3/2)t} + 1}$

10.  $\sin(2t) dt + \cos(3y) dy = 0$

SOLUTION:  $\frac{-1}{2} \cos(2t) + \frac{1}{3} \sin(3y) = C$

11.  $y' = xy^2$

SOLUTION:  $y = \frac{1}{-(1/2)x^2 - C}$

12.  $\frac{dy}{dx} - \frac{3}{2x}y = \frac{2x}{y}$  (Hint: Let  $v = y^2$ )

SOLUTION: Multiply by  $y$  to get

$$yy' - \frac{3}{2x}y^2 = 2x$$

So (following what we did for the general Bernoulli eqn), let  $v = y^2$ , and therefore  $v' = 2yy'$ . Multiply by 2 to get the right form, then substitute

$$2yy' - \frac{3}{x}y^2 = 4x \quad \Rightarrow \quad v' - \frac{3}{x}v = 4x$$

Now it is a standard linear DE. Solving, we get  $v = -4x^3 + Cx^3$ , and

$$y^2 = -4x^3 + Cx^3$$

(We'll leave in implicit form).

13.  $yy'' = (y')^2$  With the hint, we need to find a substitution for  $y''$  (and then leave the equation as a DE with unknown function  $p$  with variable  $y$ ):

$$\frac{dy}{dt} = p(y) \quad \Rightarrow \quad \frac{d^2y}{dt^2} = \frac{dp}{dy} \cdot \frac{dy}{dt} = \frac{dp}{dy}p(y)$$

Now, substitute in the expressions and divide by  $yp(y)$ :

$$y \frac{dp}{dy} p(y) = (p(y))^2 \quad \Rightarrow \quad \frac{dp}{dy} = \frac{1}{y} p(y) \quad \Rightarrow \quad \int \frac{1}{p} dp = \int \frac{1}{y} dy \quad \Rightarrow \quad \ln |p| = \ln |y| + C \quad \Rightarrow \quad p = Ay$$

where  $A$  is a constant of integration. Now, substitute again with  $p = y'$ , and we get:

$$y' = Ay \quad \Rightarrow \quad y = Pe^{At}$$

where  $P$  is another constant of integration.

We can verify our answer as well:  $y' = APe^{At}$ , and  $y'' = A^2Pe^{At}$  so that

$$yy'' = Pe^{At} \cdot A^2Pe^{At} = A^2P^2e^{2At}$$

and this expression is also  $(y')^2$ .

14.  $y' + 2y = g(t)$  with  $y(0) = 0$  and  $g(t) = 1$  on  $0 \leq t \leq 1$  and zero elsewhere.

SOLUTION: This is similar to Exercise 33, Section 2.4. In this case, we go ahead and solve starting at time 0:

$$y' + 2y = 1 \quad \Rightarrow \quad y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

and this is valid for  $0 \leq t \leq 1$ . When we hit  $t = 1$ , the dynamics change to:

$$y' + 2y = 0 \quad \Rightarrow \quad y(t) = Pe^{-2t}$$

Now we will typically choose the constants so that  $y$  is continuous. Therefore, using our previous function,  $y(1) = (1 - e^{-2})/2$ , and our current function:  $y(1) = Pe^{-2}$ , or

$$P = \frac{e^2 - 1}{2}$$

Therefore, the overall solution to the DE would be:

$$y(t) = \begin{cases} (1 - e^{-2t})/2 & \text{if } 0 \leq t \leq 1 \\ ((e^2 - 1)/2)e^{-2t} & \text{if } t > 1 \end{cases}$$

Just for fun, the direction field and solution curve are plotted in Figure 1.

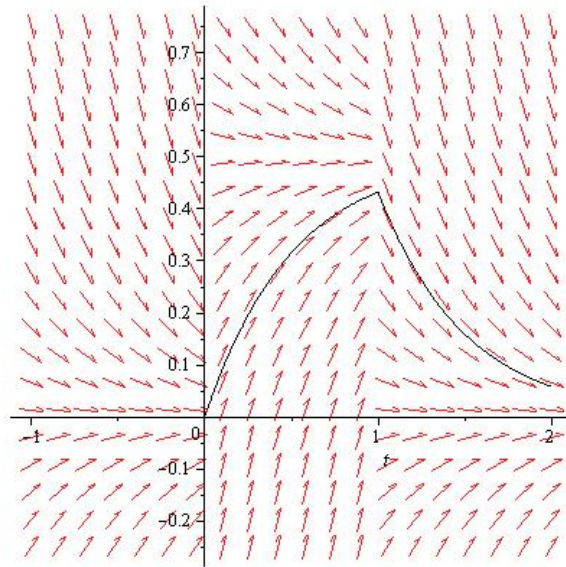


Figure 1: Direction field and solution curve for Exercise 14. Note how the solution approaches one equilibrium until  $g(t)$  changes, then it goes to the new equilibrium.

### Misc.

1. Given  $M(x, y) + N(x, y) \frac{dy}{dx} = 0$ , and assuming the equation is not currently exact, what differential equation would an integrating factor  $\mu(x)$  solve, if  $\mu$  depends on  $x$  alone?

SOLUTION: Let  $\mu(M + Ny') = 0$ . To be exact, we must have

$$\mu_x M + \mu M_x = \mu_y N + \mu N_y$$

If further,  $\mu(x)$  depends only on  $x$ , then this simplifies to:

$$\mu' M + \mu M_x = 0 + \mu N_y \quad \Rightarrow \quad \mu' = \mu \cdot \frac{N_y - M_x}{M}$$

where the fraction is an expression of  $x$  alone, so this would be a separable DE in  $\mu$ .

2. Construct a linear first order differential equation whose general solution is given by:

(a)  $y(t) = t - 3 + \frac{C}{t^2}$

SOLUTION: Construct  $y'$ . The idea will be to produce a linear DE. Therefore, we need to construct  $y'$  and compare it to  $y$ :

$$y' = 1 - 2Ct^{-3}$$

Add this to some multiple ( $t$ 's are allowed) of  $y$  to get of the arbitrary constant. In this case,

$$y' + \frac{2}{t}y = (1 - 2Ct^{-3}) + 2 - \frac{6}{t} + 2Ct^{-3} = 3 - \frac{6}{t}$$

or,

$$ty' + 2y = 3t - 6$$

(b)  $y(t) = 2 \sin(3t) + Ce^{-2t}$

SOLUTION: Same idea as before. Try to get  $y$  and  $y'$  together in such a way as to cancel out the arbitrary  $C$ :

$$y' = 6 \cos(3t) - 2Ce^{-2t}$$

so that:  $y' + 2y = 4 \sin(3t) + 6 \cos(3t)$ .

3. Suppose we want to construct a population model so that there is a logistic population growth-

- (a) That is, there is an environmental carrying capacity of 100. Construct an appropriate (autonomous) model.

SOLUTION: In the  $(y, y')$  plane, we are looking at an upside down parabola that goes through  $y = 0$  and  $y = 100$ . One model is therefore

$$y' = y(100 - y)$$

- (b) Using your model, now assume there is a continuous "harvest" (of  $k$  units per time period). How does that effect the model- In particular, is there a critical value of  $k$  over which the population will be extinct? If so, find it.

SOLUTION: This subtracts  $k$  from our population:

$$y' = y(100 - y) - k$$

We see that for some value of  $k$ , the vertex of the upside parabola is exactly on the  $y$ -axis. We can solve this by completing the square- That is, we should be able to write  $y'$  as a perfect square:

$$-y^2 + 100y + k \rightarrow -(y^2 - 100y + 50^2) = -(y - 50)^2$$

Therefore,  $k = 50^2 = 2500$ . This is the maximum allowable harvest before the ecosystem collapses.

4. We want to construct a new population model. In this case, we have a "minimum population" constraint in that, if the population every goes below this number (call it a constant  $R_1$ ), then there is not enough population to recover- The population goes extinct. There is also a larger number that we called the environmental threshold,  $R_2$ .

We want to find the simplest model that will have the desired behavior. First, noting that it is autonomous, graph  $y' = f(y)$ , then write down a possible polynomial for  $f$ .

SOLUTION: Create a cubic polynomial (3 equilibria) with zeros at  $y = 0, R_1, R_2$ . Also, consider the arrows along the  $y$ -axis indicating increasing or decreasing, and we see that the polynomial will having a leading  $-ky^3$  (with  $k > 0$ ):

$$y' = -ky(y - R_1)(y - R_2)$$

5. Suppose we have a tank that contains  $M$  gallons of water, in which there is  $Q_0$  pounds of salt. Liquid is pouring into the tank at a concentration of  $r$  pounds per gallon, and at a rate of  $\gamma$  gallons per minute. The well mixed solution leaves the tank at a rate of  $\gamma$  gallons per minute.

Write the initial value problem that describes the amount of salt in the tank at time  $t$ , and solve:

$$\frac{dQ}{dt} = r\gamma - \frac{\gamma}{M}Q, \quad Q(0) = Q_0$$

The solution is:

$$Q = rM + (Q_0 - rM)e^{-(\gamma/M)t}$$

6. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on  $Q_0$ ? Does this make sense?

Note that the differential equation for  $Q$  is autonomous, so we could do a phase plot (line with a negative slope). Or, we can just take the limit as  $t \rightarrow \infty$  and see that  $Q \rightarrow rM$ . This does not necessarily depend on  $Q_0$ ; if  $Q_0$  starts at equilibrium,  $rM$ , then  $Q$  is constant.

It does make sense. The incoming concentration of salt is  $r$  pounds per gallon, so we would expect the long term concentration to be the same,  $rM/M = r$ .

7. Modify problem 5 if:  $M = 100$  gallons,  $r = 2$  and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if  $Q_0 = 50$ .

$$\frac{dQ}{dt} = 4 - \frac{3}{100-t}Q \quad Q(0) = 50$$

This goes from being autonomous to linear. In this case, use an integrating factor,

$$e^{\int p(t) dt} = e^3 \int \frac{1}{100-t} dt = e^{-3 \ln |100-t|} = (100+t)^{-3} \quad t > 100$$

Going back to the DE

$$\left( \frac{Q}{(100-t)^3} \right)' = 4(100-t)^{-3} \Rightarrow Q = 2(100-t) + C(100-t)^3$$

Continuing, we get:

$$Q(t) = 2(100-t) - \frac{150}{100^3}(100-t)^3$$

8. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of  $\frac{1}{2}v$ , find the initial value problem (and solve it) for the velocity at time  $t$ .

The general model is:  $mv' = mg - kv$ . In this case,  $m = 1$ ,  $g = 9.8$  and  $k = 1/2$ . Therefore,

$$v' = 9.8 - \frac{1}{2}v$$

Which is linear (and autonomous). Since the object is being dropped, the initial velocity is zero.

Solve it:

$$v(t) = 19.6 \left( 1 - e^{-(1/2)t} \right)$$

9. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of  $k$  dollars per month, set up a model for how much you owe at time  $t$  in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

SOLUTION: First, notice that the units of time are mixed- The interest rate is an annual rate, but  $k$  is in dollars per month. We should first decide on what the units of time should be. The answer below will assume that we are working with time in *years*, so that the annual payments are  $12k$  dollars per year (in a continuous fashion).

Therefore, the model is  $S' = rS - 12k$ , where  $S$  will be the amount owing,  $r$  is the annual interest rate and  $k$  is the rate for the continuous payment (per month). Then using  $S(0) = S_0$ , we can write the solution as

$$S(t) = \frac{12k}{r} + \left( S_0 - \frac{12k}{r} \right) e^{rt}$$

Substituting in the values for  $r$  and  $S_0$ , and  $t = 10$ , we can solve the equation for  $k$ :

$$0 = 240k + (10000 - 240k)e^{\frac{1}{2}}$$

*Extra:* If we go ahead and solve, we get a monthly payment rate of  $k \approx \$105.90$ .

10. Show that the IVP  $xy' = y - 1$ ,  $y(0) = 2$  has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution).

SOLUTION: We'll try to solve it, and see what happens. The DE is separable. Notice that the function  $f(x, y)$  from the E& U theorem is  $(y - 1)/x$ , which is not continuous at  $x = 0$ ...

Continuing as usual:

$$\int \frac{1}{y-1} dy = \int \frac{1}{x} dx \Rightarrow \ln|y-1| = \ln|x| + C$$

Exponentiate both sides to get

$$y - 1 = Ax \Rightarrow y = Ax + 1$$

There is no choice of  $A$  that will satisfy the initial condition,

$$2 = A \cdot 0 + 1$$

11. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is  $20^2$ ), and that one year later, the population is 625 (which is  $25^2$ ). Determine the time in which the population reaches 10000 (which is  $100^2$ )

SOLUTION: If  $P(t)$  is the population at time  $t$ , then the first part of the statement translates to:

$$\frac{dP}{dt} = kP^{1/2}$$

where  $k$  is the constant of proportionality. This is separable (and autonomous), so:

$$\int P^{-1/2} dP = \int k dt \Rightarrow P^{1/2} = (k/2)t + C$$

Given the initial population, we can solve for  $C$ , given the second piece of info, we can solve for  $k$ :

$$P(0) = 20^2 \Rightarrow 20 = (k/2)(0) + C \Rightarrow C = 20$$

and  $P(1) = 25^2$  gives:

$$25 = (k/2) + 20 \Rightarrow k = 10$$

Therefore, our model is:

$$P(t) = (5t + 20)^2$$

Solving for the last part,  $P(t) = 100^2$ , we have

$$100 = 5t + 20 \Rightarrow t = 16$$

12. Consider the sketch below of  $F(y)$ , and the differential equation  $y' = F(y)$ .

- (a) Find and classify the equilibrium.

SOLUTION: From the sketch given,  $y = 0$  is asymptotically stable and  $y = 1$  is semistable.

- (b) Find intervals (in  $y$ ) on which  $y(t)$  is concave up.

SOLUTION: Examine the intervals  $y < 0$ ,  $0 < y < 1/3$ ,  $1/3 < y < 1$  and  $y > 1$  separately. The function  $y$  will be concave up when  $dF/dy$  and  $F$  both have the same sign- This happens when  $F$  is either increasing and positive (which happens nowhere) or decreasing and negative:

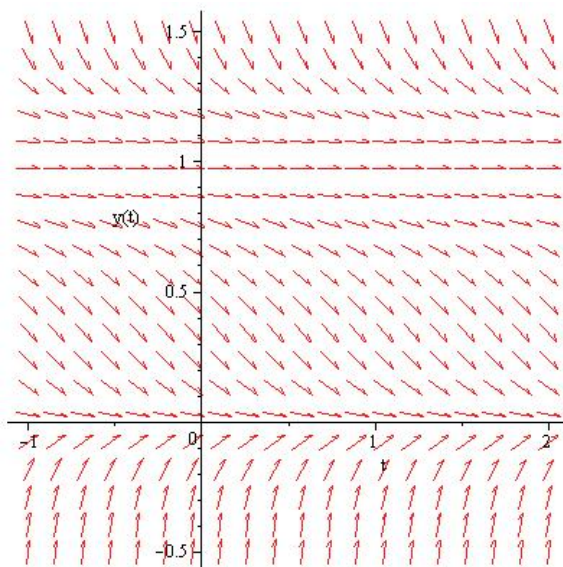
$$0 < y < \frac{1}{3} \quad y > 1$$

- (c) Draw a sketch of  $y$  on the direction field, paying particular attention to where  $y$  is increasing/decreasing and concave up/down. See the figure below.

- (d) Find an appropriate polynomial for  $F(y)$ .

SOLUTION: One example is

$$y' = -y(y-1)^2$$



13. Consider the DE:  $y dx + (2x - ye^y) dy = 0$ . Show that the equation is not exact, but becomes exact if you assume there is an integrating factor in terms of  $y$  alone (Hint: Find the integrating factor first).

SOLUTION: Let  $\mu$  be the integrating factor, and assume  $\mu$  is a function of  $y$  alone. Then

$$(\mu M)_y = \mu' \cdot M + \mu \cdot M_y = y\mu' + \mu$$

And

$$(\mu N)_x = 0 \cdot N + \mu \cdot 2$$

Setting these equal, we have the DE:

$$y\mu' = \mu \Rightarrow \int \frac{1}{\mu} d\mu = \int \frac{dy}{y}$$

or  $\mu = y$ . We can verify our answer, since  $y^2 dx + (2xy - y^2 e^y) dy = 0$  should now be exact.

14. Newton's Law of Cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the environment (which we assume is some constant). Write down a differential equation which represents this statement, then find the general solution.

SOLUTION: If  $u(t)$  is the temperature of the body at time  $t$ , and  $T$  is the (constant) environmental temperature (please define your terms!), then we have:

$$u' = -k(u - T) \Rightarrow u(t) = Ce^{-kt} + T$$

This solution makes sense, since the long term behavior is that the temperature reaches  $T$  as  $t \rightarrow \infty$ .

15. Given the direction field below, find a differential equation that is consistent with it.

SOLUTION: Draw the corresponding figure in the  $(y, y')$  plane first. There we see that  $y = 0$  is unstable (make it a linear crossing), and  $y = 2$  is stable,  $y = 4$  is unstable. From the figure,

$$y' = y(y - 2)(y - 4)$$

will work.

16. Consider the direction field below, and answer the following questions:

- (a) Is the DE possibly of the form  $y' = f(t)$ ?

SOLUTION: No. The isoclines would be vertical (consider, for example, a vertical line at  $t = -3$ ; the slopes are clearly not equal).



(b) Is the DE possible of the form  $y' = f(y)$ ?

SOLUTION: No. The isoclines would be horizontal (for example, look at a horizontal line at  $y = 1$ - Some slopes are zero, others are not).

(c) Is there an equilibrium solution? (If so, state it):

SOLUTION: Yes- At  $y = 0$ .

(d) Draw the solution corresponding to  $y(-1) = 1$ .

SOLUTION: Just draw a curve consistent with the arrows shown.

17. Evaluate the following integrals:

SOLUTIONS: You should be able to integrate by parts and use partial fractions fairly quickly at this point. Try checking your answers using Maple!

Some notes: The first integral should be done using a table, and the last should simplify a lot.

$$\int x^3 e^{2x} dx \quad \int \frac{x}{(x-1)(2-x)} dx \quad e^{-3} \int dt/t$$