

Review Questions, Exam 1, Math 244

These questions are presented to give you an idea of the variety and style of question that will be on the exam. It is not meant to be exhaustive, so be sure that you understand the homework problems.

Discussion Questions:

1. True or False, and explain:
 - (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.
 - (b) Let $\frac{dy}{dt} = 1 + y^2$. The Existence and Uniqueness theorem tells us that the solution (for any initial value) will be valid for all t . (If true, say why. If False, solve the DE).
 - (c) If $y' = \cos(y)$, then the solutions are periodic.
 - (d) All autonomous equations are separable.
 - (e) All separable equations are exact.
2. State the Existence and Uniqueness Theorem (for both linear and nonlinear IVPs)
3. Let $y' = y^{1/3}$, $y(0) = 0$. Find two solutions to the IVP. Does this violate the Existence and Uniqueness Theorem?

Solve:

Give the general solution if there is no initial value. Before giving the solution, state what kind of differential equation it is (linear, separable, exact)- Multiple classes are possible, just give the one you will use.

1. $\frac{dy}{dx} = \frac{x^2 - 2y}{x}$
2. $(x + y) dx - (x - y) dy = 0$. Let $v = y/x$, and make the substitution first to get an ODE in $v(x)$.
3. $\frac{dy}{dx} = \frac{2x + y}{3 + 3y^2 - x}$ $y(0) = 0$.
4. $\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}$
5. $\frac{dy}{dt} = 2 \cos(3t)$ $y(0) = 2$
6. $y' - \frac{1}{2}y = 0$ $y(0) = 200$. Additionally, give the interval on which the solution is valid.
7. $y' = (1 - 2x)y^2$ $y(0) = -1/6$. Additionally, give the interval on which the solution is valid.
8. $y' - \frac{1}{2}y = e^{2t}$ $y(0) = 1$

9. $y' = \frac{1}{2}y(3 - y)$
10. $\sin(2t) dt + \cos(3y) dy = 0$
11. $y' = xy^2$
12. $\frac{dy}{dx} - \frac{3}{2x}y = \frac{2x}{y}$. Let $v = y^2$ and substitute to get an ODE in $v(x)$.
13. $yy'' = (y')^2$ (Hint: Let $p(y) = y'$, and see if you can get the equation in terms of p and y).
14. $y' + 2y = g(t)$, $y(0) = 0$ and

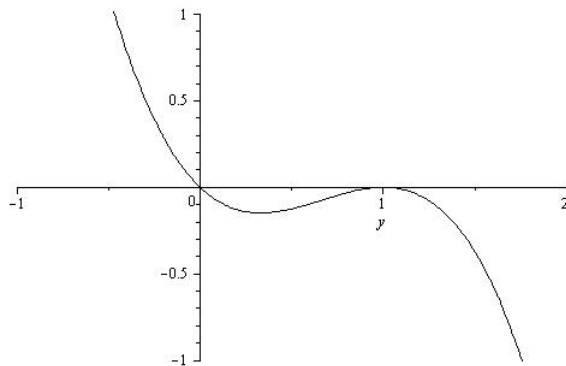
$$g(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$$

Misc.

1. Given $M(x, y) + N(x, y)\frac{dy}{dx} = 0$, and assuming the equation is not currently exact, what differential equation would an integrating factor $\mu(x)$ solve, if μ depends on x alone?
2. Construct a linear first order differential equation whose general solution is given by:
 - (a) $y(t) = t - 3 + \frac{C}{t^2}$
 - (b) $y(t) = 2 \sin(3t) + Ce^{-2t}$
3. Suppose we want to construct a population model so that there is a logistic population growth-
 - (a) That is, there is an environmental carrying capacity of 100. Construct an appropriate (autonomous) model.
 - (b) Using your model, now assume there is a continuous “harvest” (of k units per time period). How does that effect the model- In particular, is there a critical value of k over which the population will be extinct? If so, find it.
4. We want to construct a new population model. In this case, we have a “minimum population” constraint in that, if the population every goes below this number (call it a constant R_1), then there is not enough population to recover- The population goes extinct. There is also a larger number that we called the environmental threshold, R_2 . We want to find the simplest model that will have the desired behavior. First, noting that it is autonomous, graph $y' = f(y)$, then write down a possible polynomial for f .
5. Suppose we have a tank that contains M gallons of water, in which there is Q_0 pounds of salt. Liquid is pouring into the tank at a concentration of r pounds per gallon, and at a rate of γ gallons per minute. The well mixed solution leaves the tank at a rate of γ gallons per minute.

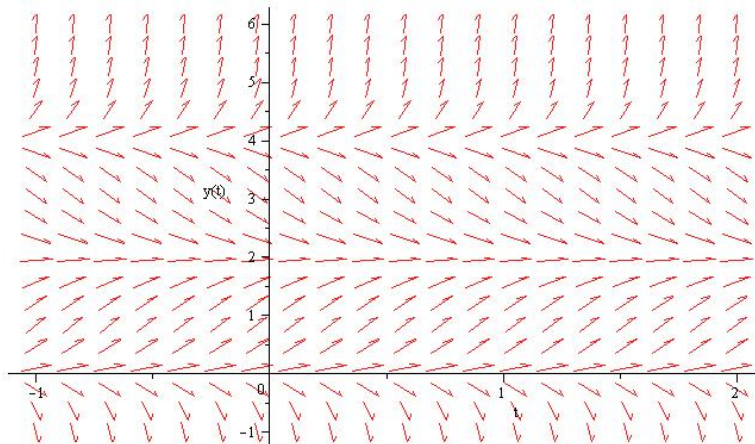
Write the initial value problem that describes the amount of salt in the tank at time t , and solve:

6. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on Q_0 ? Does this make sense?
7. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.
8. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time t . In the (t, y) plane, draw several solution curves (Hint: This is an autonomous DE).
9. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.
10. Show that the IVP $xy' = y - 1$, $y(0) = 2$ has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution).
11. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is 20^2), and that one year later, the population is 625 (which is 25^2). Determine the time in which the population reaches 10000 (which is 100^2).
12. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.
 - (a) Find and classify the equilibrium.
 - (b) Find intervals (in y) on which $y(t)$ is concave up.
 - (c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down.
 - (d) Find an appropriate polynomial for $F(y)$.

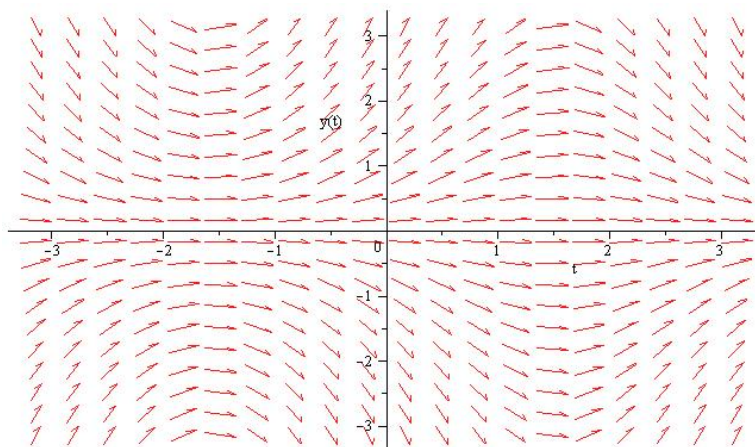


13. Consider the DE: $y dx + (2x - ye^y) dy = 0$. Show that the equation is not exact, but becomes exact if you assume there is an integrating factor in terms of y alone (Hint: Find the integrating factor first).

14. Newton's Law of Cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the environment (which we assume is some constant). Write down a differential equation which represents this statement, then find the general solution.
15. Given the direction field below, find a differential equation that is consistent with it.



16. Consider the direction field below, and answer the following questions:
- Is the DE possibly of the form $y' = f(t)$?
 - Is the DE possible of the form $y' = f(y)$?
 - Is there an equilibrium solution? (If so, state it):
 - Draw the solution corresponding to $y(-1) = 1$.



17. Evaluate the following integrals:

$$\int x^3 e^{2x} dx \quad \int \frac{x}{(x-1)(2-x)} dx \quad e^{-3 \int dt/t}$$