Exercise Set for Complex Eigenvalues

1. (a) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

   SOLUTION: $Tr(A) = 2, \det(A) = -3 + 8 = 5$, so the characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$. Solving for λ, we get $\lambda = 1 \pm 2i$.

   Using $\lambda = 1 + 2i$, solve the system of equations (we only need the first equation):

$$\begin{array}{ll} (a-\lambda)v_1 + bv_2 & = 0 \\ cv_1 + (d-\lambda)v_2 & = 0 \end{array} \quad \Rightarrow \quad (2-2i)v_1 - 2v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2-2i \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

   The solution to the DE is found by first computing $e^{\lambda t}\mathbf{v}$:

$$e^{(1+2i)t}\begin{bmatrix} 1 \\ 1-i \end{bmatrix} = e^t(\cos(2t) + i\sin(2t))\begin{bmatrix} 1 \\ 1-i \end{bmatrix} =$$

$$e^t\begin{bmatrix} \cos(2t) + i\sin(2t) \\ (\cos(2t) + \sin(2t)) + i(-\cos(2t) + \sin(2t)) \end{bmatrix}$$

   From this, the solution is

$$\mathbf{x}(t) = C_1\text{real}(e^{\lambda t}\mathbf{v}) + C_2\text{imag}(e^{\lambda t}\mathbf{v})$$

$$\mathbf{x}(t) = e^t\left[C_1\begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} + C_2\begin{bmatrix} \sin(2t) \\ -\cos(2t) + \sin(2t) \end{bmatrix}\right]$$

   The origin in this case is a "spiral source". We should find that the rotation is CCW.

   (b) $\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$

   SOLUTION: $Tr(A) = 0, \det(A) = 1$, so the characteristic equation is $\lambda^2 + 1 = 0$, so $\lambda = \pm i$.

   Using $\lambda = i$, solve the system of equations (we only need the first equation):

$$\begin{array}{ll} (a-\lambda)v_1 + bv_2 & = 0 \\ cv_1 + (d-\lambda)v_2 & = 0 \end{array} \quad \Rightarrow \quad (2-i)v_1 - 5v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$$

   The solution to the DE is found by first computing $e^{\lambda t}\mathbf{v}$:

$$e^{it}\begin{bmatrix} 5 \\ 2-i \end{bmatrix} = (\cos(t) + i\sin(t))\begin{bmatrix} 5 \\ 2-i \end{bmatrix} =$$

$$\begin{bmatrix} 5\cos(t) + i5\sin(t) \\ (2\cos(t) + \sin(t)) + i(-\cos(t) + 2\sin(t)) \end{bmatrix}$$

From this, the solution is

$$\mathbf{x}(t) = C_1\text{real}(e^{\lambda t}\mathbf{v}) + C_2\text{imag}(e^{\lambda t}\mathbf{v})$$

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + C_2 \begin{bmatrix} 5\sin(2t) \\ -\cos(t) + 2\sin(t) \end{bmatrix}$$

Because there is no real exponential function, the origin will be a "center" (solutions are periodic), and the rotation is CCW.

(c) (Oops! Same as (a))

2. (a) Given $\lambda = -1 + 2i$, and $\mathbf{v} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$, the solution is

$$\mathbf{x}(t) = C_1\text{real}(e^{\lambda t}\mathbf{v}) + C_2\text{imag}(e^{\lambda t}\mathbf{v})$$

$$e^{(-1+2i)t} \begin{bmatrix} 1-i \\ 2 \end{bmatrix} = e^{-t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1-i \\ 2 \end{bmatrix} =$$

$$e^{-t} \begin{bmatrix} (\cos(2t) + \sin(2t)) + i(-\cos(2t) + \sin(2t) \\ 2\cos(2t) + i2\sin(2t) \end{bmatrix}$$

Therefore,

$$\mathbf{x}(t) = e^{-t} \left[C_1 \begin{bmatrix} \cos(2t) + \sin(2t) \\ 2\cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} -\cos(2t) + \sin(2t) \\ 2\sin(2t) \end{bmatrix} \right]$$

The origin is a spiral sink.

(b) $\lambda = -2, 3$ with $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The solution is:

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The origin is a saddle.

(c) $\lambda = -1, -3$ with $v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

The solution is:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The origin is a sink.

(d) Given $\lambda = 1 + 3i$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$, the solution is

$$\mathbf{x}(t) = C_1 \text{real}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2 \text{imag}(\mathrm{e}^{\lambda t}\mathbf{v})$$

$$\mathrm{e}^{(1+3i)t} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = \mathrm{e}^t (\cos(3t) + i \sin(3t)) \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} =$$
$$\mathrm{e}^t \begin{bmatrix} \cos(3t) + i \sin(3t) \\ (\cos(3t) + \sin(3t)) + i(-\cos(3t) + \sin(3t) \end{bmatrix}$$

Therefore,

$$\mathbf{x}(t) = \mathrm{e}^t \left[C_1 \begin{bmatrix} \cos(3t) \\ \cos(3t) + \sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(3t) \\ -\cos(3t) + \sin(3t) \end{bmatrix} \right]$$

The origin is a spiral source.

(e) Given $\lambda = 2i$, and $\mathbf{v} = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$, the solution is

$$\mathbf{x}(t) = C_1 \text{real}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2 \text{imag}(\mathrm{e}^{\lambda t}\mathbf{v})$$

$$\mathrm{e}^{2it} \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} = (\cos(2t) + i \sin(2t)) \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} =$$
$$\begin{bmatrix} (\cos(2t) - \sin(2t)) + i(\cos(2t) + \sin(2t) \\ \cos(2t) + i \sin(2t) \end{bmatrix}$$

Therefore,

$$\mathbf{x}(t) = C_1 \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) \end{bmatrix} + C_1 \begin{bmatrix} \cos(2t) + \sin(2t) \\ \sin(2t) \end{bmatrix}$$

The origin is a center.