Solutions to Review Questions: Exam 3

1. What is the ansatz we use for y in

   - Chapter 6? SOLUTION: $y(t)$ is piecewise continuous and is of exponential order (so that $Y(s)$ exists).
   - Section 5.2? SOLUTION: $y(x)$ is analytic at $x = x_0$. That is,

   $$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

2. Finish the definitions:

   - The Heaviside function, $u_c(t)$:

   $$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \qquad c > 0$$

   - The Dirac $\delta-$function: $\delta(t - c)$

   $$\delta(t-c) = \lim_{\tau \to 0} d_\tau(t-c)$$

   where

   $$d_\tau(t-c) = \begin{cases} \frac{1}{2\tau} & \text{if } c - \tau < t < c + \tau \\ 0 & \text{elsewhere} \end{cases}$$

   - Define the convolution: $(f * g)(t)$

   $$(f * g)(t) = \int_0^t f(t-u) g(u)\, du$$

   - A function is of **exponential order** if:
   there are constants M, k, and a so that

   $$|f(t)| \leq M e^{kt} \quad \text{for all} \quad t \geq a$$

3. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)

   $$f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 6 - t, & t \geq 2 \end{cases}$$

   $$\int_0^\infty e^{-st} f(t)\, dt = \int_0^2 3e^{-st}\, dt + \int_2^\infty (6-t)e^{-st}\, dt$$

   The second antiderivative is found by integration by parts:

   $$\int_2^\infty (6-t)e^{-st}\, dt \Rightarrow \begin{array}{ccc} \hline + & 6-t & e^{-st} \\ - & -1 & (-1/s)e^{-st} \\ + & 0 & (1/s^2)e^{-st} \end{array} \Rightarrow \quad e^{-st}\left(-\frac{6-t}{s} + \frac{1}{s^2}\right)\Big|_2^\infty$$

   Putting it all together,

   $$-\frac{3}{s}e^{-st}\Big|_0^2 + \left(0 - e^{-2s}\left(-\frac{4}{s} + \frac{1}{s^2}\right)\right) = -\frac{3e^{-2s}}{s} + \frac{3}{s} + \frac{4e^{-2s}}{s} - \frac{e^{-2s}}{s^2} = \frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

   NOTE: Did you remember to *anti*differentiate in the third column?

(b)

$$f(t) = \begin{cases} e^{-t}, & 0 \le t < 5 \\ -1, & t \ge 5 \end{cases}$$

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^5 e^{-st}e^{-t}\,dt + \int_5^\infty -e^{-st}\,dt = \int_0^5 e^{-(s+1)t}\,dt + \int_5^\infty -e^{-st}\,dt$$

Taking the antiderivatives,

$$-\frac{1}{s+1}e^{-(s+1)t}\Big|_0^5 + \frac{1}{s}e^{-st}\Big|_5^\infty = \frac{1}{s+1} - \frac{e^{-5(s+1)}}{s+1} + 0 - \frac{e^{-5s}}{s}$$

4. Check your answers to Problem 3 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

(a) $f(t) = 3(u_0(t) - u_2(t)) + (6-t)u_2(t) = 3 - 3u_2(t) + (6-t)u_2(t) = 3 + (3-t)\,u_2(t)$
For the second term, notice that the table entry is for $u_c(t)h(t-c)$. Therefore, if

$$h(t-2) = 3 - t \quad \text{then} \quad h(t) = 3 - (t+2) = 1 - t \quad \text{and} \quad H(s) = \frac{1}{s} - \frac{1}{s^2}$$

Therefore, the overall transform is:

$$\frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

(b) $f(t) = e^{-t}\,(u_0(t) - u_5(t)) - u_5(t)$
We can rewrite f in preparation for the transform:

$$f(t) = e^{-t}u_0(t) - e^{-t}u_5(t) - u_5(t)$$

For the middle term,

$$h(t-5) = e^{-t} \quad \Rightarrow \quad h(t) = e^{-(t+5)} = e^{-5}e^{-t}$$

so the overall transform is:

$$F(s) = \frac{1}{s+1} - e^{-5}\frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s}$$

5. Show that $\cos(t)$ is of exponential order.

SOLUTION: Recall that $f(t)$ is of exponential order means that $|f(t)| \le Me^{kt}$ for $t \ge c$. Since $\cos(t) \le 1$ for all t, then we can take $M = 1$ and $k = 0$ (or $k = 1$).

$$|\cos(t)| \le 1 = e^{0t} \text{ for } t \ge 0$$

6. Write the following functions in piecewise form (thus removing the Heaviside function):

(a) $(t+2)u_2(t) + \sin(t)u_3(t) - (t+2)u_4(t)$

SOLUTION: First, notice that $(t+2)$ is turned "on" at time 2. At time $t=3$, $\sin(t)$ joins the first function, and at time $t=4$, we subtract the function $t+2$ back off.

$$\begin{cases} 0 & \text{if } 0 \le t < 2 \\ t+2 & \text{if } 2 \le t < 3 \\ t+2+\sin(t) & \text{if } 3 \le t < 4 \\ \sin(t) & \text{if } t \ge 4 \end{cases}$$

(b) $\sum_{n=1}^{4} u_{n\pi}(t)\sin(t-n\pi)$

SOLUTION: First (you can determine this graphically) $\sin(t-\pi) = -\sin(t)$, and $\sin(t-2\pi) = \sin(t)$, and $\sin(t-3\pi) = -\sin(t)$, etc.- You should simplify these. Therefore:

$$\begin{cases} 0 & \text{if } 0 \le t < \pi \\ \sin(t-\pi) & \text{if } \pi \le t < 2\pi \\ \sin(t-\pi)+\sin(t-2\pi) & \text{if } 2\pi < t < 3\pi \\ \sin(t-\pi)+\sin(t-2\pi)+\sin(t-3\pi) & \text{if } 3\pi \le t < 4\pi \\ \sin(t-\pi)+\sin(t-2\pi)+\sin(t-3\pi)+\sin(t-4\pi) & \text{if } t \ge 4\pi \end{cases} =$$

$$\begin{cases} 0 & \text{if } 0 \le t < \pi \\ -\sin(t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi < t < 3\pi \\ -\sin(t) & \text{if } 3\pi \le t < 4\pi \\ 0 & \text{if } t \ge 4\pi \end{cases}$$

7. Determine the Laplace transform:

(a) $t^2 e^{-9t}$

$$\frac{2}{(s+9)^3}$$

(b) $e^{2t} - t^3 - \sin(5t)$

$$\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$$

(c) $t^2 y'(t)$. Use the table, $\mathcal{L}(-t^n f(t)) = F^{(n)}(s)$. In this case, $F(s) = sY(s) - y(0)$, so $F'(s) = sY'(s) + Y(s)$ and $F''(s) = sY''(s) + 2Y'(s)$.

(d) $e^{3t}\sin(4t)$

$$\frac{4}{(s-3)^2+16}$$

(e) $e^t\delta(t-3)$

In this case, notice that $f(t)\delta(t-c)$ is the same as $f(c)\delta(t-c)$, since the delta function is zero everywhere except at $t=c$. Therefore,

$$\mathcal{L}(e^t\delta(t-c)) = e^3 e^{-3s}$$

(f) $t^2 u_4(t)$

In this case, let $h(t-4) = t^2$, so that

$$h(t) = (t+4)^2 = t^2 + 8t + 16 \quad \Rightarrow \quad H(s) = \frac{2 + 8s + 16s^2}{s^3}$$

and the overall transform is $\mathrm{e}^{-4s} H(s)$.

8. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$

$$\frac{2s-1}{s^2-4s+6} = \frac{2s-1}{(s^2-4s+4)+2} = 2\frac{s-1/2}{(s-2)^2+2} =$$

In the numerator, make $s - \frac{1}{2}$ into $s - 2 + \frac{3}{2}$, then

$$2\left(\frac{s-2}{(s-2)^2+2} + \frac{3}{2\sqrt{2}}\frac{\sqrt{2}}{(s-2)^2+2}\right) \Rightarrow 2\mathrm{e}^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\mathrm{e}^{2t}\sin(\sqrt{2}t)$$

(b) $\dfrac{7}{(s+3)^3} = \dfrac{7}{2!}\dfrac{2!}{(s+3)^3} \Rightarrow \dfrac{7}{2}t^2\mathrm{e}^{-3t}$

(c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)} = \mathrm{e}^{-2s}H(s)$, where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{2}{s+2} \quad \Rightarrow \quad h(t) = 2\mathrm{e}^{t} + 2\mathrm{e}^{-2t}$$

and the overall inverse: $u_2(t)h(t-2)$.

(d) $\dfrac{3s-1}{2s^2-8s+14}$ Complete the square in the denominator, factoring the constants out:

$$\frac{3s-1}{2(s^2-8s+5)} = \frac{3}{2}\cdot\frac{s-1/3}{(s-2)^2+3} = \frac{3}{2}\left(\frac{s-2}{(s-2)^2+3} + \frac{5}{3}\cdot\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s-2)^2+3}\right)$$

The inverse transform is:

$$\frac{3}{2}\mathrm{e}^{2t}\cos(\sqrt{3}t) + \frac{5}{2\sqrt{3}}\mathrm{e}^{2t}\sin(\sqrt{3}t)$$

(e) $\left(\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right)\dfrac{1}{s^2+s-6} = \left(\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right)H(s)$

Where:

$$H(s) = \frac{1}{s^2+s-6} = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3}$$

so that

$$h(t) = \frac{1}{5}\mathrm{e}^{2t} - \frac{1}{5}\mathrm{e}^{-3t}$$

and the overall transform is:

$$u_2(t)h(t-2) - u_3(t)h(t-3)$$

9. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

   (a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

$$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}$$

   so that

$$Y(s) = \frac{2}{s^3(s^2+2s+2)} + \frac{4}{s^2(s^2+2s+2)} - \frac{1}{s^2+2s+2}$$

   (b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$

$$s^2Y + s - 5 + 9Y = \frac{10}{s-2} \Rightarrow Y(s) = \frac{10}{(s-2)(s^2+9)} - \frac{s-5}{s^2+9}$$

   (c) $y'' - 4y' + 4y = t^2e^t$, $y(0) = 0$, $y'(0) = 0$

$$(s^2 - 4s + 4)Y = \frac{2}{(s-1)^3} \Rightarrow Y(s) = \frac{2}{(s-1)^3(s-2)^2}$$

10. Solve the given initial value problems using Laplace transforms:

   (a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.

$$Y = \frac{e^{-5s}}{2s^2+s+2} = e^{-5s}H(s)$$

   where

$$H(s) = \frac{1}{2s^2+s+2} = \frac{1}{2}\frac{1}{s^2+\frac{1}{2}s+1} = \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\frac{4}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

   Therefore,

$$h(t) = \frac{2}{\sqrt{15}}e^{-1/4\,t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

   And the overall solution is $u_5(t)h(t-5)$

   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$

$$s^2Y + 3s - 10 + 6(sY+3) + 9Y = 0 \quad \Rightarrow \quad Y = -\frac{3s+8}{(s+3)^2}$$

   Partial Fractions:

$$-\frac{3s+8}{(s+3)^2} = -\frac{3}{(s+3)} + \frac{1}{(s+3)^2} \Rightarrow y(t) = -3e^{-3t} + te^{-3t}$$

(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$

$$Y = \mathrm{e}^{-s}\frac{1}{s(s-3)(s+1)} + \frac{1}{(s+1)(s-3)} = \mathrm{e}^{-s}H(s) + \frac{1}{4}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s+1}$$

where

$$H(s) = \frac{1}{s(s-3)(s+1)} = -\frac{1}{3}\frac{1}{s} + \frac{1}{12}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}$$

so that

$$h(t) = -\frac{1}{3} + \frac{1}{12}\mathrm{e}^{3t} + \frac{1}{4}\mathrm{e}^{-t}$$

and the overall solution is:

$$y(t) = \frac{1}{4}\mathrm{e}^{3t} - \frac{1}{4}\mathrm{e}^{-t} + u_1(t)h(t-1)$$

(d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$

$$Y = \mathrm{e}^{-\pi/2\, s}\frac{1}{s^2+4} + \frac{1}{s^2+4}$$

Therefore,

$$y(t) = \frac{1}{2}\sin(2t) + u_{\pi/2}(t)\frac{1}{2}\sin(2(t-\pi/2))$$

(e) $y'' + y = \sum_{k=1}^{\infty}\delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

$$Y(s) = \sum_{k=1}^{\infty}\mathrm{e}^{-2k\pi s}\frac{1}{s^2+1}$$

Therefore, term-by-term,

$$y(t) = \sum_{k=1}^{\infty}u_{2k\pi}(t)\sin(t - 2\pi k) = \sum_{k=1}^{\infty}u_{2\pi k}(t)\sin(t)$$

Piecewise,

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 2\pi \\ \sin(t) & \text{if} \quad 2\pi \le t < 4\pi \\ 2\sin(t) & \text{if} \quad 4\pi \le t < 6\pi \\ 3\sin(t) & \text{if} \quad 6\pi \le t < 8\pi \\ \vdots & \vdots \end{cases}$$

11. For the following, use Laplace transforms to solve, and leave your answer in the form of a convolution:

(a) $4y'' + 4y' + 17y = g(t) \qquad y(0) = 0, y'(0) = 0$
SOLUTION: First, note that

$$4s^2 + 4s + 17 = 4(s^2 + s + 17/4) = 4((s + 1/2)^2 + 4)$$

Therefore,

$$Y(s) = \frac{G(s)}{4s^2+4s+17} = G(s)\cdot\frac{1}{8}\frac{2}{(s+\frac{1}{2})^2+2^2}$$

Therefore,

$$y(t) = g(t) * \frac{1}{8}\mathrm{e}^{-t/2}\sin(2t)$$

(b) $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t)$, with $y(0) = 1$ and $y'(0) = -1$.
SOLUTION: Take the Laplace transform of both sides:

$$(s^2Y - s + 1) + (sY - 1) + \frac{5}{4}Y = \frac{1}{s} - \frac{\mathrm{e}^{-\pi s}}{s}$$

so that

$$Y(s) = \frac{1-\mathrm{e}^{-\pi s}}{s(s^2+s+5/4)} + \frac{s}{s^2+s+5/4}$$

For the second term,

$$\frac{s}{s^2+s+5/4} = \frac{s}{(s+\frac{1}{2})^2+1} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1} - \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+1}$$

For the first term, treat it like:

$$\left(\mathrm{e}^{-0s} - \mathrm{e}^{-\pi s}\right)H(s)$$

where

$$H(s) = \frac{1}{s}\cdot\frac{1}{s^2+s+\frac{5}{4}} = \frac{1}{s}\cdot\frac{1}{(s+\frac{1}{2})^2+1}$$

so that

$$h(t) = 1 * \mathrm{e}^{-t/2}\sin(t)$$

Therefore, the overall answer is:

$$y(t) = h(t) - u_\pi(t)h(t-\pi) + \mathrm{e}^{-t/2}\left(\cos(t) - \frac{1}{2}\sin(t)\right)$$

12. Short Answer:

(a) $\int_0^\infty \sin(3t)\delta(t-\frac{\pi}{2})\,dt = \sin(3\pi/2) = -1$, since

$$\int_0^\infty f(t)\delta(t-c)\,dt = f(c)$$

(b) Use Laplace transforms to solve the first order DE, thus finding which function has the Dirac function as its derivative:

$$y'(t) = \delta(t-c), \qquad y(0) = 0$$

SOLUTION:

$$sY = \mathrm{e}^{-cs} \Rightarrow Y = \frac{\mathrm{e}^{-cs}}{s}$$

so that $y(t) = u_c(t)$. Therefore, the "derivative" of the Heaviside function is the Dirac $\delta-$function!

(c) Use Laplace transforms to solve for $F(s)$, if

$$f(t) + 2\int_0^t \cos(t-x) f(x)\, dx = \mathrm{e}^{-t}$$

(So only solve for the transform of $f(t)$, don't invert it back).

$$F(s) + 2F(s)\frac{s}{s^2+1} = \frac{1}{s+1} \qquad \Rightarrow \qquad F(s)\left(\frac{(s+1)^2}{s^2+1}\right) = \frac{1}{s+1}$$

so that

$$F(s) = \frac{s^2+1}{(s+1)^3}$$

(d) In order for the Laplace transform of f to exist, f must be?
f must be piecewise continuous and of exponential order

13. Let $f(t) = t$ and $g(t) = u_2(t)$.

(a) Use the Laplace transform to compute $f * g$.
To use the table,

$$\mathcal{L}(t * u_2(t)) = \frac{1}{s^2} \cdot \frac{\mathrm{e}^{-2s}}{s} = \mathrm{e}^{-2s}\frac{1}{s^3} = \mathrm{e}^{-2s}H(s)$$

so that $h(t) = \frac{1}{2}t^2$. The inverse transform is then

$$u_2(t)\frac{1}{2}(t-2)^2$$

(b) Verify your answer by directly computing the integral.
By direct computation, we'll choose to "flip and shift" the function t:

$$f * g = \int_0^t (t-x)u_2(x)\, dx$$

Notice that $u_2(x)$ is zero until $x = 2$, then $u_2(x) = 1$. Therefore, if $t \leq 2$, the integral is zero. If $t \geq 2$, then:

$$\int_0^t (t-x)u_2(x)\, dx = \int_2^t t - x\, dx = tx - \frac{1}{2}x^2\Big|_2^t = t^2 - \frac{1}{2}t^2 - 2t + 2 = \frac{1}{2}(t-2)^2$$

valid for $t \geq 2$, zero before that. This means that the convolution is:

$$t * u_2(t) = \frac{1}{2}(t-2)^2 u_2(t)$$

14. If $a_0 = 1$, determine the coefficients a_n so that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Try to identify the series represented by $\sum_{n=0}^{\infty} a_n x^n$.

SOLUTION: The recognition problem is a little difficult, but we should be able to get the coefficients:

$$\sum_{k=0}^{\infty} \left[(k+1)a_{k+1} + 2a_k\right] x^k = 0 \qquad \Rightarrow \qquad a_{k+1} = -\frac{2}{k+1} a_k$$

Just doing the straight computations, we get:

$$y(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \cdots$$

To see the pattern, it is easiest to look at the general terms (Typically, I wouldn't ask for the recognition part on the exam, but you should be able to get the first few computations, as we did above):

$$\begin{array}{lll} a_1 & = -2a_0 & = \frac{(-2)}{1!} a_0 \\ a_2 & = -\frac{2}{2} a_0 = 2a_0 & = \frac{4}{2!} a_0 \\ a_3 & -\frac{2}{3} a_2 = -\frac{4}{3} a_0 & = \frac{-8}{3!} a_0 \\ \vdots & & \vdots \end{array}$$

The series is for $\mathrm{e}^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{n!}$

15. Write the following as a single sum in the form $\sum_{k=2}^{\infty} c_k (x-1)^k$ (with a few terms in the front):

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + x(x-2)\sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

In front of the second sum we have $x^2 - 2x$, but we can't bring that directly into the sum since we have powers of $(x-1)$. But, we might recognize that:

$$x^2 - 2x = (x^2 - 2x + 1) - 1 = (x-1)^2 - 1$$

Therefore, the second sum can be expanded into two sums:

$$((x-1)^2 - 1)\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = (x-1)^2 \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} =$$

$$\sum_{n=1}^{\infty} na_n(x-1)^{n+1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

Now we have three sums to work with

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n+1} - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

In the first sum, the first non-zero term has $(x-1)^0$, the second sum begins with $(x-1)^2$, and the last sum starts with $(x-1)^0$. We could shift the second index to start at $n = 0$, but then the sum begins with $(x-1)^1$. We'll have to break off the constant terms from the first two sums:

$$\sum_{n=1}^{\infty} n(n-1)a_n(x-1)^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1)(x-1)^{n-2}$$

and similarly
$$-\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = -a_1 - \sum_{n=2}^{\infty} na_n(x-1)^{n-1}$$
Now we can bring all three sums together. In the first sum, we'll substitute $k = n-2$ (or $n = k+2$). In the middle sum, $k = n+1$ (or $n = k-1$), and in the third sum, $k = n-1$ (or $n = k+1$). With these substitutions, we get:

$$2a_2 - a_0 + \sum_{k=1}^{\infty} \left((k+2)(k+1)a_{k+2} + (k-1)a_{k-1} - (k+1)a_{k+1}\right)(x-1)^k$$

NOTE: The question asked for the index to start at $k=2$ instead of $k=1$- It's OK to do it either way; mainly, I wanted to see you put the sums together as one.

16. Characterize ALL (continuous or not) solutions to

$$y'' + 4y = u_1(t), \qquad y(0) = 1, y'(0) = 1$$

SOLUTION: The idea behind this question is to get you to think about the kinds of solutions we get from the Laplace transform. If we do not require y to be continuous, then this DE is actually two differential equations:

$$y'' + 4y = 0, \qquad y(0) = 1, y'(0) = 1 \qquad \text{valid for } t \le 1$$

And

$$y'' + 4y = 1 \qquad y(1), y'(1) \text{ arbitrary }, \text{ valid for } t > 1$$

The general solution is then:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \le 1 \\ c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

If we require $y(t)$ to be continuous (a very common assumption), then we get the answer that comes from using Laplace transforms. Writing the answer in piecewise form:

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t) & \text{if } t \le 1 \\ -\frac{1}{4}\cos(2(t-1)) + \frac{1}{4} & \text{if } t > 1 \end{cases}$$

17. Use the table to find an expression for $\mathcal{L}(ty')$. Use this to convert the following DE into a linear first order DE in $Y(s)$ (do not solve):

$$y'' + 3ty' - 6y = 1, y(0) = 0, y'(0) = 0$$

SOLUTION: For the first part, use Table Entry 15. In particular,

$$\mathcal{L}(tf(t)) = -F'(s)$$

where, in our case, $f(t) = y'(t)$, so that $F(s) = sY - y(0)$. Therefore,

$$\mathcal{L}(tf(t)) = -(Y - sY') = sY' - Y$$

Substituting this into the DE, we get:

$$Y' + \left(\frac{s^2 - 3s - 6}{3s}\right) Y = \frac{1}{s}$$

18. Find the recurrence relation between the coefficients for the power series solutions to the following:

   (a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.
   Substituting our power series in for y, y', y'':

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

   We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

   Noting that in the second sum we could start at $n = 0$, since the first term (constant term) would be zero anyway, we can start all series with a constant term:

$$\sum_{k=0}^{\infty} \left(2(k+2)(k+1)a_{k+2} + ka_k + 3a_k\right) x^k = 0$$

   From which we get the recurrence relation:

$$a_{k+2} = -\frac{k+3}{2(k+2)(k+1)}\, a_k$$

   (b) $(1-x)y'' + xy' - y = 0$, $x_0 = 0$
   Substituting our power series in for y, y', y'':

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

   We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

   The two middle sums can have their respective index taken down by one (so that formally the series would start with $0x^0$):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

   Now make all the indices the same. To do this, in the first sum make $k = n-2$, in the second sum take $k = n-1$. Doing this and collecting terms:

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - (k+1)ka_{k+1} + (k-1)a_k\right) x^k = 0$$

   So we get the recursion:

$$a_{k+2} = \frac{(k+1)k\, a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

(c) $y'' - xy' - y = 0$, $x_0 = 1$

SOLUTION: Let $y = \sum_{n=0}^{\infty} a_n (x-1)^n$ so that

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Substituting these into the differential equation, we get

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - x\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

We need to bring the x into the sum, but we can only do that if we had $x - 1$. Therefore, we re-write x as: $x = (x-1) + 1$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - [(x-1)+1]\sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

The middle term is rewritten as:

$$-[(x-1)+1]\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = -\sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

Incorporating this sum into the whole gives us four sums now:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=1}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

The second sum needs to begin at $n = 0$ for our powers to all begin with $(x-1)^0$:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=0}^{\infty} na_n(x-1)^n - \sum_{n=1}^{\infty} na_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

In the first sum, take $m = n - 2$ (or $n = m + 2$), in the second sum, $m = n$ and in the third sum, $m = n - 1$, and in the fourth sum, $m = n$. Writing all sums in terms of m, we get:

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m - \sum_{m=0}^{\infty} ma_m(x-1)^m - \sum_{m=1}^{\infty}(m+1)a_{m+1}(x-1)^m - \sum_{m=0}^{\infty} a_m(x-1)^m = 0$$

Write this as a single sum (I simplified the first and third together):

$$\sum +m = 0^{\infty} \left((m+2)(m+1)a_{m+2} - (m+1)a_m - (m+1)a_{m+1}\right)(x-1)^m = 0$$

Now we get the recurrence relation (you can write it using any index you like):

$$a_{n+2} = \frac{1}{n+2}\left(a_{n+1} + a_n\right), \qquad \text{for } n = 0, 1, 2, 3, \cdots$$

19. Exercises with the table:

(a) SOLUTION: Prove formula #6 using 4 and 11)

$$\mathcal{L}(e^{at}\sin(bt)) = F(s-a)$$

where

$$F(s) = \mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2} \qquad \Rightarrow \qquad \frac{b}{(s-a)^2+b^2}$$

Therefore,

$$\mathcal{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

(b) Show that you can use table entry 15 to find the Laplace transform of $t^2\delta(t-3)$ (verify your answer using a property of the δ function).

SOLUTION: Using Entry 11, the Laplace transform of $t^2\delta(t-3)$ is the second derivative of the Laplace transform of $\delta(t-3)$. That is, using

$$F(s) = e^{-3s}$$

then

$$\mathcal{L}(t^2\delta(t-3)) = F''(s) = 9e^{-3s}$$

And this is the same as:

$$\int_0^\infty e^{-st}t^2\delta(t-3)\,dt = 9e^{-3s}$$

(c) Prove (using the definition of $\mathcal{L}$) table entries 9, 10

SOLUTION: 9 is a special case of 10, so we prove 10 using the definition:

$$\mathcal{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st}f(t-c)\,dt$$

We want this answer to be the following (with a different variable of integration):

$$e^{-cs}F(s) = e^{-cs}\int_0^\infty e^{-sw}f(w)\,dw = \int_0^\infty e^{-s(w+c)}f(w)\,dw$$

We can connect the two by taking $w = t-c$ (so that $t = w+c$), and then (remember to change the bounds!):

$$\int_c^\infty e^{-st}f(t-c)\,dt = \int_0^\infty e^{-s(w+c)}f(w)\,dw$$

And we're done.

(d) Prove (using the definition of $\mathcal{L}$) a formula (similar to 14) for $\mathcal{L}(y'''(t))$.

SOLUTION: I wanted you to recall how we got those definitions in the past (integrating by parts):

$$\mathcal{L}(y'''(t)) = \int_0^\infty e^{-st}y'''(t)\,dt$$

Integration by parts using a table:

$$\begin{array}{rrr} + & e^{-st} & y'''(t) \\ - & -se^{-st} & y''(t) \\ + & s^2e^{-st} & y'(t) \\ - & -s^3e^{-st} & y(t) \end{array} \quad \Rightarrow \quad \left(e^{-st}\left(y''(t)+sy'(t)+s^2y(t)\right)\Big|_{t=0}^{\infty} + s^3\int_0^{\infty} e^{-st}y(t)\,dt\right.$$

At infinity, these terms all go to zero (otherwise, the Laplace transform wouldn't exist), so we get:

$$s^3 - (y''(0) + sy'(0) + s^2y(0)) = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

20. Find the first 5 terms of the power series solution to $e^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

Compute the derivatives directly, then (don't forget to divide by $n!$):

$$y(x) = 1 - x - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 + \ldots$$

21. Find the radius of convergence for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}x^n$

SOLUTION:

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}}|x| = |x|$$

So by the ratio test, the series will converge (absolutely) if $|x| < 1$ (so the radius is 1).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n$

SOLUTION: Simplifying the limit in the ratio test, we get

$$\lim_{n\to\infty} 2\sqrt{\frac{n}{n+1}}|x+3| = 2|x+3|$$

Therefore, by the ratio test, the series will converge absolutely if $2|x+3| < 1$, or if $|x+3| < 1/2$ (and this is our radius). For the interval of convergence, we have to check the points $x = -7/2$ and $x = -5/2$ separately. For $x = -7/2$, the series diverges ($p-$test), and for $x = -5/2$, the series converges by the alternating series test.

NOTE: If you don't recall those tests, you probably ought to review them, but I won't make you recall them for the exam this week.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$

SOLUTION: For the Ratio Test, first simplify the ratio:

$$\frac{(n+1)^2 3^n}{n^2 3^{n+1}}|x+2| = \left(\frac{n+1}{n}\right)^2 \frac{|x+2|}{3}$$

The limit is $|x+2|/3$, so the radius of convergence is 3. For extra practice, we can also find the interval of convergence. We need to test the endpoints:
For $x = -2 - 3 = -5$: The series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (-3)^n}{3^n} = \sum_{n=1}^{\infty} n^2$$

which is divergent. A similar computation shows divergence at $x = -2 + 3 = 1$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n5^n}$

SOLUTION: The Ratio Test simplifies to:

$$\frac{1}{5} \lim_{n\to\infty} \frac{n}{n+1} |3x-2| = \frac{|3x-2|}{5}$$

To converge absolutely, $|3x-2| < 5$. To get the radius of convergence, we need to have the form $|x-a| < \rho$, so in this case, we simplify to get:

$$3\left|x - \frac{2}{3}\right| < 5 \quad \Rightarrow \quad \left|x - \frac{2}{3}\right| < \frac{5}{3}$$

Now we have to check the endpoints separately, which are $x = -1$ and $x = 7/3$:

- At $x = -1$, the sum becomes:

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

  This is an alternating harmonic series, which converges (but not absolutely).
- At $x = 7/3$, the sum becomes a harmonic series, which diverges.

The interval of convergence is: $[-1, \frac{7}{3})$