M244
Sample 01

Final Exam

Name:_____

Show all your work! You may not use your text, colleagues or a calculator. A table of Laplace transforms and the Poincare Diagram is provided.

1. Short Answer:
   (a) True or False (give a short reason): The solution to $y' = f(y)$ may be periodic.
   (b) Evaluate the integral: $\int_0^5 \delta(t-2) + 3u_4(t)\,dt$
   (c) Given the system below, convert to a second order DE (do not solve the DE):
   $$\begin{aligned} x_1' &= 2x_1 + x_2 \\ x_2' &= x_1 + 2x_2 \end{aligned}$$
2. Short Answer
   (a) Solve: $y'' + 4y' + 4y = 0$
   (b) Solve: $y'' + 2y' + 5y = 0$
   (c) Label each of the previous two problems as: Underdamped, Overdamped, or Critically Damped.
3. Give your (final) ansatz for the particular solution to the following using the Method of Undetermined Coefficients. Do NOT actually find the coefficients!
   (a) $y' - 5y = t^2$
   (b) $y'' - 2y' + 5y = te^t \sin(2t)$
4. Consider the system: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \alpha & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
   Using the Poincaré Diagram (you might sketch it), describe how changing α changes the classification of the origin.
5. Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$
6. Consider the system: $\begin{aligned} x' &= 2x + 6y \\ y' &= 2x - 2y \end{aligned}$
   (a) Find the general solution.
   (b) Find the particular solution with initial value $(1, 3)$.
7. Laplace Transforms:

(a) Find the Laplace transform of the solution, $Y(s)$ (do **not** solve for $y(t)$!)

$$y''-2y'+y = g(t) \qquad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } g(t) = \begin{cases} t/2 & \text{if } 0 \le t < 2 \\ 1 & \text{if } t \ge 2 \end{cases}$$

(b) Compute the inverse Laplace transform of $\dfrac{\mathrm{e}^{-3s}(s+1)}{s^2+4s+6}$

8. A tank originally contains 100 gallons of fresh water. Water containing 3/5 lb of salt per gallon is poured into the tank at a rate of 5 gal/min, and the mixture is pumped out at 7 gal/min. Write an IVP modeling the amount of salt at time t (do not solve it!).

9. Find the general solution: $\dfrac{dy}{dt} - \dfrac{2}{t}y = t^3\mathrm{e}^t$

10. Find the radius of convergence for the power series: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

11. Write the power series of the solution, $y(x)$, centered at $x_0 = 0$, up to and including the x^5 term (use the derivatives of y directly).

$$y'' - 2x^2y' + y = 0 \qquad y(0) = 1, \quad y'(0) = 1$$