# HW to Replace 7.1: Systems of Differential Equations

Try them all, but turn in solutions to 1(a), 3(a) and all of 6.

1. For each second order differential equation below, write the corresponding system of first order equations. Lastly, write the system in matrix-vector form.

   (a) $y'' + 2y' - 3y = 0$

   (b) $y'' + 4y' + 4y = 0$

   (c) $y'' - 9y = 0$

   (d) $y'' - 2y' + 2y = 0$

2. For extra practice, go back and give the general solution to each second order DE (using methods from Chapter 3 of our textbook).

3. For each system of first order, convert to a corresponding second order differential equation.

   (a) $\begin{aligned} x' &= 3x - 2y \\ y' &= 2x - 2y \end{aligned}$

   (b) $\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned}$

   (c) $\begin{aligned} x' &= x + y \\ y' &= 4x - 2y \end{aligned}$

4. For extra practice, solve each of the second order DEs from the previous exercise using techniques from Chapter 3 of our text.

5. Consider the system below:

$$\begin{aligned} x' &= -3x + y \\ y' &= -2y \end{aligned}$$

   Solve this system by recognizing that we can solve for $y$ directly, then substitute this into the DE for $x$ and solve it as a first order linear DE.

6. Each system below is *nonlinear*. Solve each by first writing the system as $dy/dx$.

   (a) $\begin{aligned} x' &= y(1 + x^3) \\ y' &= x^2 \end{aligned}$

   (b) $\begin{aligned} x' &= 4 + y^3 \\ y' &= 4x - x^3 \end{aligned}$

   (c) $\begin{aligned} x' &= 2x^2y + 2x \\ y' &= -(2xy^2 + 2y) \end{aligned}$

7. Convert the third order (nonlinear) differential equation into a system of first order equations.

$$y''' - y'' + y'y = t^2$$

# SOLUTIONS

1. For each second order differential equation below, write the corresponding system of first order equations.

(a) $y'' + 2y' - 3y = 0$

SOLUTION: Let $u = y, v = y'$. Then notice that $v' = y'' = 3y - 2y' = 3u - 2v$. The full system is then:

$$\begin{aligned} u' &= v \\ v' &= 3u - 2v \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

NOTE: Always write your variables in the same order. Since we started with $u, v$, then $u'$ came first, then $v'$, etc.

(b) $y'' + 4y' + 4y = 0$

SOLUTION: Let $u = y, v = y'$. Then $v' = y'' = -4y - 4y' = -4u - 4v$, and

$$\begin{aligned} u' &= v \\ v' &= -4u - 4v \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

(c) $y'' + 9y = 0$

SOLUTION: Let $u = y, v = y'$. Then $v' = y'' = -9y = -9u$, and

$$\begin{aligned} u' &= v \\ v' &= -9u \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

(d) $y'' - 2y' + 2y = 0$

SOLUTION: Let $u = y, v = y'$. Then $v' = y'' = 2y' - 2y$, and

$$\begin{aligned} u' &= v \\ v' &= -2u + 2v \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

2. For extra practice, go back and give the general solution to each second order DE (using methods from Chapter 3 of our textbook).

SOLUTIONS:

(a) $y'' + 2y' - 3y = 0$

The characteristic equation is $r^2 + 2r - 3 = 0$, or $(r + 3)(r - 1) = 0$, so $r = 1, -3$. Therefore, the general solution is

$$y(t) = C_1 e^t + C_2 e^{-3t}$$

(b) $y'' + 4y' + 4y = 0$

The characteristic equation is $r^2 + 4r + 4 = 0$, or $(r + 2)^2 = 0$, so $r = -2, -2$. Therefore, the general solution is

$$y(t) = e^{-2t}(C_1 + C_2 t)$$

(c) $y'' + 9y = 0$

The characteristic equation is $r^2 + 9 = 0$, or $r = \pm 3i$,

$$y(t) = C_1 \cos(3t) + C_2 \sin(3t)$$

(d) $y - 2y' + 2y = 0$

The characteristic equation is $r^2 - 2r + 2 = 0$, or $r = 1 \pm i$, so $r = 1, -3$. Therefore, the general solution is

$$y(t) = \mathrm{e}^t (C_1 \cos(t) + C_2 \sin(t))$$

3. For each system of first order, convert to a corresponding second order differential equation.

(a) $\begin{aligned} x' &= 3x - 2y \\ y' &= 2x - 2y \end{aligned}$

SOLUTION: Using the first equation to solve for $y$, $y = -(1/2)(x' - 3x)$. Putting this into the second equation:

$$\left(-\frac{x' - 3x}{2}\right)' = 2x - 2\left(-\frac{x' - 3x}{2}\right)$$

Clean up by multiplying both sides by $-2$:

$$x'' - 3x' = -4x - 2(x' - 3x) \quad \Rightarrow \quad x'' - x' - 2x = 0$$

(b) $\begin{aligned} x' &= -2x + y \\ y' &= x - 2y \end{aligned}$

SOLUTION: Using the first equation to solve for $y$, $y = x' + 2x$. Putting this into the second equation:

$$(x' + 2x)' = x - 2\,(x' + 2x) \quad \Rightarrow \quad x'' + 4x' + 3x = 0$$

(c) $\begin{aligned} x' &= x + y \\ y' &= 4x - 2y \end{aligned}$

SOLUTION: Similar to the others, $y = x' - x$ from equation 1, so equation 2 becomes:

$$x'' - x' = 4x - 2(x' - x) \quad \Rightarrow \quad x'' + x' - 6x = 0$$

4. For extra practice, solve each of the second order DEs from the previous exercise using techniques from Chapter 3 of our text.

(a) $x(t) = C_1 e^{2t} + C_2 e^{-t}$ (b) $x(t) = C_1 e^{-t} + C_2 e^{-3t}$ (c) $x(t) = C_1 e^{2t} + C_2 e^{-3t}$

5. Consider the system below:
$$\begin{aligned} x' &= -3x + y \\ y' &= -2y \end{aligned}$$
Solve this system by recognizing that we can solve for $y$ directly, then substitute this into the DE for $x$ and solve it as a first order linear DE.

SOLUTION: Looking at $y' = -2y$, $y(t) = C_1 e^{-2t}$. Putting that into the equation for $x'$, we get:
$$x' + 3x = C_1 e^{-2t}$$
The integrating factor is $e^{3t}$, so that
$$(xe^{3t})' = C_1 e^t \quad \Rightarrow \quad xe^{-3t} = C_1 e^t + C_2 \quad \Rightarrow \quad x(t) = C_1 e^{-2t} + C_2 e^{-3t}$$
As a parametric system, we have:
$$\begin{bmatrix} C_1 e^{-2t} + C_2 e^{-3t} \\ C_1 e^{-2t} \end{bmatrix}$$
We would note that the $C_1$ for $y$ is the SAME as the $C_1$ for the $x$.

6. Each system below is *nonlinear*. Solve each by first writing the system as $dy/dx$.

(a) $\begin{aligned} x' &= y(1 + x^3) \\ y' &= x^2 \end{aligned}$

SOLUTION: This becomes separable.
$$\frac{dy}{dx} = \frac{x^2}{y(1 + x^3)} \quad \Rightarrow \quad \int y\, dy = \int \frac{x^2\, dx}{1 + x^3} \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}\ln(1 + x^3) + C$$
At this point, we'll leave it since we won't know if we need the positive or negative root.

(b) $\begin{aligned} x' &= 4 + y^3 \\ y' &= 4x - x^3 \end{aligned}$

SOLUTION: This becomes separable:
$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \quad \Rightarrow \quad \int 4 + y^3\, dy = \int 4x - x^3\, dx \quad \Rightarrow \quad 4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C$$
We'll leave this in implicit form.

(c) $\begin{aligned} x' &= 2x^2 y + 2x \\ y' &= -(2xy^2 + 2y) \end{aligned}$

SOLUTION: This becomes exact (don't cancel anything!)

$$\frac{dy}{dx} = -\frac{2xy^2 + 2y}{2x^2y + 2x} \quad \Rightarrow \quad (2xy^2 + 2y) + (2x^2y + 2x)\frac{dy}{dx} = 0$$

This is exact, with $M_y = N_x = 4xy + 2$. The solution is

$$x^2y^2 + 2xy = C$$

7. Convert the third order (nonlinear) differential equation into a system of first order equations.

$$y''' - y'' + y'y = t^2$$

SOLUTION: Let $x_1 = y, x_2 = y'$, and $x_3 = y''$. Then:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_1x_2 + x_3 + t^2 \end{aligned}$$