Homework for 3.7

1. For practice with trig, 3.7.1-3.7.4, also given below. In each case, write the sum as $R\cos(\omega t - \delta)$

   3.7.1 $3\cos(2t) + 4\sin(2t)$
   SOLUTION: For each of these, think of $A\cos(2t)+B\sin(2t)$ as defining a complex number $A+iB$. Then R is the magnitude and δ is the angle for $A+iB$. In this particular case, we see that (A,B) is in Quadrant I, so δ does not need an extra π added to it:

$$R = \sqrt{9+16} = 5 \qquad \delta = \tan^{-1}(4/3) \Rightarrow$$
$$3\cos(2t) + 4\sin(2t) = 5\cos(2t - \tan^{-1}(4/3))$$

   3.7.2 $-\cos(t) + \sqrt{3}\sin(t)$
   SOLUTION: Note that in this case, $(-1,\sqrt{3})$ is in Quadrant II, so add π to δ. Also, notice that the angle δ is coming from a triangle with side $1, 2, \sqrt{3}$ (or 30-60-90). In this case,

$$R = \sqrt{1+3} = \sqrt{2} \qquad \delta = \tan^{-1}(-\sqrt{3}) = -\pi/3$$
$$-\cos(t) + \sqrt{3}\sin(t) = 2\cos(t - (2\pi/3))$$

   3.7.3 $4\cos(3t) - 2\sin(3t)$
   SOLUTION: In this case, $(4,-2)$ is coming from Quadrant IV, so no need to add π to δ. We don't have a special triangle in this case.

$$R = \sqrt{16+4} = 2\sqrt{5} \qquad \delta = \tan^{-1}(-1/2)$$
$$4\cos(3t) - 2\sin(3t) = 2\sqrt{5}\cos(t - \tan^{-1}(-1/2))$$

   3.7.4 $-2\cos(\pi t) - 3\sin(\pi t)$
   SOLUTION: In this case $(-2,-3)$ is in Quadrant III, so we'll need to add π to δ.

$$R = \sqrt{4+9} = \sqrt{13} \qquad \delta = \tan^{-1}\left(\frac{3}{2}\right) + \pi$$

2. Practice with the Model (metric system):

   (a) A spring with a 3-kg mass is held stretched 0.6 meters beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the position of the mass after t seconds (assume no damping).

SOLUTION: $m = 3$, $\gamma = 0$, so we just need the spring constant k. By Hooke's Law, the force is proportional to the length stretched:

$$k(0.6) = 20 \quad \Rightarrow \quad k = \frac{100}{3}$$

Now we have the IVP:

$$3u'' + \frac{100}{3}u = 0 \qquad u(0) = 0, \quad u'(0) = \frac{6}{5}$$

To solve this,

$$3r^2 + \frac{100}{3} = 0 \qquad r = \sqrt{\frac{100}{9}}\,i = \frac{10}{3}\,i$$

The general solution is

$$C_1 \cos\left(\frac{10}{3}\,t\right) + C_2 \sin\left(\frac{10}{3}\,t\right)$$

Putting in the initial conditions,

(b) A spring with a 4-kg mass has a natural length of 1 meter, and is maintained stretched to a length of 1.3 m by a force of 24.3 N. If the spring is compressed to a length of 0.8 m and then released with zero velocity, find the position of the mass after t seconds (assume no damping).

SOLUTION: $m = 4$, $\gamma = 0$, and the wording of the question implies that the spring is stretched 0.3 m beyond natural length, giving (convert to fractions)

$$\frac{3k}{10} = \frac{243}{10} \quad \Rightarrow \quad k = 81$$

To set up the IVP, compressing the spring to a length of 0.8 m means that the initial position will be -0.2, or $-1/5$. Therefore, we have (recall that "up" is negative)

$$4u'' + 81u = 0 \qquad u(0) = -\frac{1}{5} \quad u'(0) = 0$$

Now solve $u'' + \frac{81}{4}u = 0$:

$$u(t) = C_1 \cos\left(\frac{9}{2}\,t\right) + C_2 \sin\left(\frac{9}{2}\,t\right))$$

Solving for the constants, you should get $C_1 = -1/5, C_2 = 0$.

(c) A spring with a mass of 2 kg has damping constant 14 and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is then stretched to 1 m beyond its natural length and released. Find the position of the mass at any time t.

SOLUTION: $m = 2$, $\gamma = 14$ and for the spring constant, we have:

$$\frac{k}{2} = 6 \quad \Rightarrow \quad k = 12$$

The IVP is: $2u'' + 14u' + 12u = 0$, with $u(0) = 1$ and $u'(0) = 0$

$$u(t) = -\frac{1}{5}\mathrm{e}^{-6t} + \frac{6}{5}\mathrm{e}^{-t}$$

(d) Using the approximation $g \approx 49/5$, suppose the mass is 20/49, and the mass displaces the spring an additional 1/2 meter. The mass is in a medium that exerts a viscous resistance of 6 N when the velocity is 3 m/s. If the mass is pulled down an additional 1/2 m and released, formulate an IVP that models the motion of the mass.

SOLUTION: From what is given, we use the fact that at the equilibrium ($u(t) = 0$), $mg - kL = 0$. In that case,

$$mg = kL \quad \Rightarrow \quad \frac{20}{49}\frac{49}{5} = k \cdot \frac{1}{2} \quad \Rightarrow \quad k = 8$$

For the damping constant, recall that the force due to damping is assumed to be proportional to velocity, so that

$$F = \gamma u'(t) \quad \Rightarrow \quad 6 = \gamma 3 \quad \Rightarrow \quad \gamma = 2$$

Therefore, the IVP is given by:

$$\frac{20}{49}u'' + 2u' + 8u = 0 \qquad u(0) = \frac{1}{2} \quad u'(0) = 0$$

(e) Using $g \approx 49/5$, if the weight of the mass, $mg = 2$, and attached to the spring it stretches the spring 1/2 m, determine the spring constant and mass. If the spring is compressed (from equilibrium) by 1/10 m, and then released, and if there is no damping, determine the IVP that governs the motion of the mass.

SOLUTION: We aren't given a force for the spring, so we use equilibrium (so that $mg - kL = 0$).

$$2 - k\frac{1}{2} = 0 \quad \Rightarrow \quad k = 4$$

Since $mg = 2$, we'll need to solve for the mass:

$$m\frac{49}{5} = 2 \quad \Rightarrow \quad m = \frac{10}{49}$$

Then the spring equation with initial conditions is given by:

$$\frac{10}{49}u'' + 4u = 0 \quad u(0) = -\frac{1}{10} \quad u'(0) = 0$$

(f) For the spring model with a mass of 4 kg and a spring constant of 123, find the damping constant that would produce critical damping.

SOLUTION:

$$4u'' + \gamma u' + 123u = 0$$

For critical damping, $\gamma^2 - 4(4)(123) = 0$, so that $\gamma = 4\sqrt{123}$ (only take the positive root!)

(g) Suppose we consider a mass-spring system with no damping (the damping constant is then 0), so that the differential equation expressing the motion of the mass can be modeled as

$$mu'' + ku = 0$$

Find value(s) of β so that $A\cos(\beta t)$ and $B\sin(\beta t)$ are each solutions to the homogeneous equation (for arbitrary values of A, B).

SOLUTION: $\beta = \sqrt{\frac{k}{m}}$