In Class Example (SOLUTION): Analyze the Velocity of a Raindrop

A raindrop is falling from a motionless cloud. We would like to come up with a relationship between the relevant variables that are listed below. Use the Buckingham-π Theorem.

Name	symbol	Dim
velocity	v	LT^{-1}
radius	r	L
gravity	g	LT^{-2}
air density	ρ	ML^{-3}
air viscosity	μ	$M(LT)^{-1}$

Our process was to:

- Form a generic dimensionless quantity using combinations of the variables in the list (use π as a the generic quantity).

  $$\pi = v^a r^b g^c \rho^d \mu^e \quad \Rightarrow \quad [\pi] = [v]^a [r]^b [g]^c [\rho]^d [\mu]^e$$

  so that the dimensions:

  $$1 = (LT^{-1})^a L^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$$

  Combining terms:

  $$1 = L^{a+b+c-3d-e} M^{d+e} T^{-a-2c-e}$$

  which gives:

  $$\begin{aligned} a+b+c-3d-e &= 0 \\ d+e &= 0 \\ a+2c+e &= 0 \end{aligned}$$

  We have three equations and 5 unknowns, so two variables are free. We will choose a to be one of the free variables, since it is the exponent of velocity (we'll see why this is nice below). We can choose the other to be, for example, d.

- Form a complete set of dimensionless quantities (use the free variables to get $\pi_1, \pi_2, \ldots$)

  To get the complete set of variables, first set $a = 1, d = 0$ (then we'll set $a = 0, d = 1$). In that case,

  $$\begin{aligned} 1+b+c-0-e &= 0 \\ 0+e &= 0 \\ 1+2c+e &= 0 \end{aligned}$$

  so that $e = 0$, $c = -1/2$, $b = -1/2$. Our first dimensionless variable is:

  $$\pi_1 = vr^{-1/2}g^{-1/2} = \frac{v}{\sqrt{rg}}$$

  The second variable is found by taking $a = 0, d = 1$ so that $e = -1, c = 1/2$, and $b = 3/2$:

  $$\pi_2 = r^{3/2}g^{1/2}\rho\mu_{-1} = \frac{\rho\sqrt{gr^3}}{\mu}$$

- Use the π theorem.

  The $\pi-$Theorem now states that:

  $$f(\pi_1, \pi_2) = 0$$

or, using the Implicit Function Theorem, we can say that

$$\pi_1 = h(\pi_2)$$

From which we get:

$$\frac{v}{\sqrt{rg}} = h\left(\frac{\rho\sqrt{gr^3}}{\mu}\right)$$

NOTE: Since we could choose the exponent of velocity to be 1 and 0 respectively, that meant that v will only appear in ONE of the two variables. This allows us to solve for v as:

$$v = \sqrt{rg}\, h\left(\frac{\rho\sqrt{gr^3}}{\mu}\right)$$

In Class Example (SOLUTION): Analyze the Wind Force on a Van

Find a relationship between the force of the wind on a van driving down the highway. We'll use the following variables. The surface area refers to the area that is exposed to the wind.

Name	symbol	Dim
Force	F	MLT^{-2}
velocity	v	LT^{-1}
surface area	A	L
air density	ρ	ML^{-3}

We will use the same approach as before:

$$\pi = F^a v^b A^c \rho^d \quad \Rightarrow \quad [\pi] = [F]^a [v]^b [A]^c [\rho]^d \quad \Rightarrow \quad 1 = (MLT^{-2})^a (LT^{-1})^b L^c (ML^{-3})^d$$

so that:

$$1 = M^{a+d} L^{a+b+c-3d} T^{-2a-b}$$

so that we now have 4 unknowns and three equations. Which variable should we make the free variable? Let's make a the free variable so we can isolate F. This gives:

$$a = 1, b = -2, c = -1, d = -1$$

and we have only 1 dimensionless product:

$$\pi_1 = \frac{F}{v^2 A\rho}$$

The $\pi-$Theorem says that our relationship must be of the form:

$$f(\pi_1) = 0$$

which we will assume implies that π_1 is constant:

$$\pi_1 = k \quad \Rightarrow \quad F = kv^2 A\rho$$

NOTE: Compare this to the drag force on an airplane that we did earlier!