Exam 2 Overview Questions, Math 250

This exam will cover the linear and nonlinear programming and some neural networks questions. Be sure to review your homework problems, and be able to answer the following questions.
Please bring a calculator to the exam. The exam will be "closed book" and "closed notes".

1. True or False, and explain:
   (a) In a nonlinear program, the domain D is convex.
   (b) It is possible that a solution to a linear program is in the middle of an edge to the domain.
   (c) Let the domain D of a nonlinear program be given as $A\mathbf{x} \leq \mathbf{b}$. It is possible that a solution is in the middle of an edge of D.
   (d) The following set of points forms a convex set:
   $$\left\{ \mathbf{x} \, : \, \begin{bmatrix} 3 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right\}$$
2. Discussion/Short Answer:
   (a) State the "Fundamental Theorem of Linear Programming":
   (b) State the Extreme Value Theorem:
   (c) What is a "linear program"?
   (d) Finish the definition: A set D is convex if:
   (e) Under what conditions may we get no solution to a linear program? Does this change for a nonlinear program?
   (f) If $f : \mathbb{R}^n \to \mathbb{R}^n$, then the linearization of f at $\mathbf{x} = \mathbf{a}$ is:
   (g) If $f : \mathbb{R}^n \to \mathbb{R}$, then the linearization of f at $\mathbf{x} = \mathbf{a}$ is:
   Given the same f, the second order Taylor approximation is:
   (h) If we state that, "Given $g(x, y) = k$, it is possible under certain conditions to write this as $y = \hat{g}(x)$", what theorem are we using?
   (i) Let $\mathbf{x} \in \mathbb{R}^n$. Define the $p-$norm and $\infty-$norm for $\mathbf{x}$.
   (j) When should we use which error to compute our line of "best" fit?
   (k) What is a neural network? What is it used for?
3. Suppose our data set consists of the 3 points: $(1, 2)$, $(2, 1)$, $(3, 4)$. We want to find the line of best fit.

(a) Write (explicitly, without $\sum$) the error function using the 2−norm and find the line of best fit (by hand).

(b) Write (explicitly, without $\sum$) the error function using the 1−norm. Translate the problem into a linear programming problem (without explicit use of the ϵ_i, although you may use them to get to your final answer).

(c) Write (explicitly, without $\sum$) the error function using the ∞−norm. Translate the problem into a linear programming problem.

4. Solve the linear program graphically: $\max_{x,y}\ x + y$ so that $x \geq 0$, $y \geq 0$, $3x + 2y \leq 5$, $2x + 3y \leq 5$.

   If the objective function is replaced by $\alpha x + y$ with $\alpha > 0$, for what values of α is the solution still optimal?

5. Consider the linear program: $\max 3x + y$ such that $x, y \geq 0$, $x + y \geq 2$, $x - y \leq 2$ and $y \leq 3$.

   (a) Graph the feasible region and label all vertices of the polytope.

   (b) Solve the LP graphically.

   (c) Show that this is the solution by using the vertices.

6. Use Lagrange multipliers to find the maximum of $4x + 6y$ if $x^2 + y^2 = 16$.

7. Find the critical points of $f(x, y) = 3x - x^3 - 2y^2 + y^4$.

8. Find the global maximum of $f(x, y) = 1 + xy - x - y$ on the domain D bounded by the parabola $y = x^2$ and the line $y = 4$.

9. Recall that for a 2×2 matrix, we can compute the inverse directly:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   Use this formula to compute one step of multidimensional Newton's method, if $f(x, y)$ is given below, and the inital guess is $(1, 1)$.

$$f(x, y) = \begin{bmatrix} x^2 + 2y \\ 3xy - y^2 \end{bmatrix}$$

10. Let $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z}$. At the point $(4, 2, 1)$, find the maximum rate of change of f and the direction in which it occurs.

11. Formulate the following problem as an LP. Do not solve it:

    A confectioner manufactures two kinds of candy bars; "ProteinPlus", that has no carbohydrates, and "SugarPlus", with no fat. ProteinPlus sells for a profit of 40 cents per bar, and SugarPlus sells for a profit of 50 cents per bar. The candy is processed in three main operations; blending, cooking and packaging. The following table records the average time in minutes required by each bar for each of the processing operations:

	Blending	Cooking	Packaging
ProteinPlus	1	5	3
SugarPlus	2	4	1

    During each production run, the blending equipment is available for a maximum of 12 machine hours, the cooking equipment is available for at most 30 machine hours, and the packaging equipment for no more than 15 hours. If this machine time can be allocated to the making of either type of candy at all times that it is available, determine how many boxes of each kind the confectioner should make in order to realize the maximum profit.

12. (Referring to the previous problem) Solve the LP problem.

13. Suppose we are given a neural network with 2 input nodes, 3 nodes in the hidden layer, and 1 output node.

    (a) Our domain is in $\mathbb{R}^?$ Our range is in $\mathbb{R}^?$.

    (b) How many weights and biases are there altogether? (Assume a bias on the output node).

    (c) For the input and output layers, assume $\sigma(x) = x$, and for the hidden layer, assume $\sigma(x) = \arctan(x)$. If the weights and biases are:

    $$W^{(1)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{b}^{(1)} = \begin{bmatrix} 0.5 \\ 0.1 \\ -1 \end{bmatrix} \qquad W^{(2)} = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \qquad \mathbf{b}^{(2)} = 1$$

    compute the prestates and states of each node if the data point $(1, 1)$ is input to the network.

14. Fill in the blank with *perpendicular* or *parallel*. Also answer the associated questions.

    (a) Let $f : \mathbb{R} \to \mathbb{R}^2$. The line at $f(2)$ in the direction of $f'(2)$ is ____________ to the curve $f(t)$.

    (b) Let $z = f(x, y)$. The gradient of f at (a, b) is ____________ to a level curve of f. Show your answer is correct, if $z = f(x, y) = x^2 + 3xy$ at the point $(1, 1)$. Which level curve are you on?

(c) Let $k = f(x, y)$ be a level curve that can be written as $y = g(x)$. Then at a point (a, b), the gradient of f at (a, b) is ____________ to the line $y - b = g'(a)(x - a)$.

(d) Suppose $z = f(x, y)$ with the domain restriction that $g(x, y) = k$. If the optimum of f (with the restricted domain) is (a, b), then:

- Does $\nabla f(a, b) = 0$?
- $\nabla f(a, b)$ is ____________ to $\nabla g(a, b)$

15. Compute the gradient and Hessian of $f(x, y) = 5 - 7y + 8x^2y + 3y^2$.

16. Write the equations necessary to solve the following problem (but do not solve it):

    Suppose there are n points on the unit sphere. Find the positions of the points so that they all have maximal separation from each other. (This is from a physics problem, where the points are electrons, and they repel each other).