Homework Set 3 SOLUTIONS
Problems 2.16, 2.17, 2.18, 2.23

2.16 Previously, we determined the units of force by considering:

$$F = ma \quad \Rightarrow \quad [F] = [m][a] \quad \Rightarrow \quad [F] = M \cdot \frac{L}{T^2} = \frac{ML}{T^2}$$

Therefore, for the equation below to be dimensionally consistent, we must have:

$$F_D = kv^2 \quad \Rightarrow \quad [F_D] = [k][v]^2 \quad \Rightarrow \quad \frac{ML}{T^2} = [k]\left(\frac{L}{T}\right)^2$$

Solve for $[k]$ to get that the units of the constant must be:

$$[k] = \frac{M}{L}$$

2.17 We are given:

$$F = \frac{Gm_1m_2}{r^2}$$

For the equation to be dimensionally consistent, the units of G can be computed as:

$$[F] = \frac{[G][m_1][m_2]}{[r]^2} \quad \Rightarrow \quad \frac{ML}{T^2} = \frac{[G]M^2}{L^2} \quad \Rightarrow \quad [G] = \frac{L^3}{MT^2}$$

Note that in this case, G is not the same thing as g, the acceleration due to gravity that we were using previously.

We form dimensionless products as usual:

$$\pi = F^aG^bm_1^cm_2^dr^e \quad \Rightarrow \quad [\pi] = [F]^a[G]^b[m_1]^c[m_2]^d[r]^e$$

From these we get the following equations:

$$\begin{aligned} a + b &= 0 \\ a + 3b + e &= 0 \\ a - b + c + d &= 0 \end{aligned}$$

Since F is to the a power, we'll choose it as one of the free variables, and then:

$$b = -a, \quad e = 2a \quad c + d = b - a = -2a$$

So we can choose, for example, d to be the second free variable. To get the complete set of dimensionless quantities, we first let $a = 1, d = 0$. This gives $b = -1, c = -2, e = 2$ and

$$\pi_1 = \frac{Fr^2}{Gm_1^2}$$

Similarly, if $a = -0, d = 1$,

$$\pi_2 = \frac{m_2}{m_1}$$

So by the Buckingham $\pi-$Theorem,

$$f(\pi_1, \pi_2) = 0$$

so that the Implicit Function Theorem gives that π_1 is a function of π_2:

$$\pi_1 = h(\pi_2) \quad \Rightarrow \quad \frac{Fr^2}{Gm_1^2} = h\left(\frac{m_2}{m_1}\right)$$

In our case, we take $h(\pi_2) = \pi_2$ to get:

$$F = \frac{Gm_1^2}{r^2} \cdot \frac{m_2}{m_1} = \frac{Gm_1 m_2}{r^2}$$

2.18 If we attempt to model the pendulum without taking g into account, we should get:

$$M^a L^b 1^c T^d = 1$$

which only has the trivial solution, $a = 0, b = 0, d = 0$, c is free.

2.23 NOTES: The t in the function ψ should be P, and the drag force term used in the model will simply be κ, and not velocity.

Including the term κ, which has units ML^{-1} by Problem 2.16, we have:

m	M
l	L
g	LT^{-2}
θ	1
P	T
κ	ML^{-1}

so that the generic dimensionless product will be:

$$\pi = m^a l^b g^c \theta^d P^e \kappa^f \quad \Rightarrow \quad 1 = M^a L^b (LT^{-2})^c 1^d T^e (ML^{-1})^f$$

so that we get that d is free, and:

$$\begin{aligned} a + f &= 0 \\ b + c - f &= 0 \\ -2c + e &= 0 \end{aligned}$$

We will choose the exponent of P to be free (e is free), which leaves one other choice. The following will use a. Given the three free variables (a, d, e), we will get three dimensionless variables, each associated to (respectively), $(-1, 0, 0), (0, 1, 0), (0, 0, 1)$:

$$\pi_1 = \frac{l\kappa}{m}, \quad \pi_2 = \theta, \quad \pi_3 = \frac{Pl^{1/2}}{g^{1/2}}$$

(NOTE: Normally we use $(1,0,0)$ for π_1, but we changed it so that our variables would match those in the question). Use the Buckingham $\pi-$Theorem to get the functional form $f(\pi_1, \pi_2, \pi_3) = 0$. We will take this to mean that π_2 can be written in terms of π_1, π_3, so:

$$\theta = \psi\left(\frac{Pg^{1/2}}{l^{1/2}}, \frac{m}{l\kappa}\right)$$