An Overview: Chapters 1, 2 and Stats

1. Introductory Material: Deterministic vs Stochastic, Continuous vs Discrete, Empirical vs Analytic, Newton's Method, Difference Equations.

2. Cobweb Diagrams
   (a) Definitions: Function iteration, fixed points (equilibria).
   (b) Graphical analysis of $x_{k+1} = f(x_k)$
   (c) Graphical analysis of supply v. demand and market fluctuations.

3. Proportions
   (a) Definitions: A proportional to B, A inversely proportional to B, A is jointly proportional to B and C.
   (b) Properties:
      i. $A \propto B \Rightarrow B \propto A$.
      ii. $A \propto B$, $B \propto C \Rightarrow A \propto C$.

4. Scaling
   (a) Definitions: Geometrically similar
   (b) Linearize scaling relationships for analysis in Matlab. (e.g., If $y = Ax^n$, how do we make this linear?)
   (c) Case Study: Biological Scaling

5. Dimensional Analysis

   NOTE: I will give you the standard units for things like acceleration, velocity, g, etc.

   (a) Definitions: Dimensionally consistent (or dimensionally homogeneous), dimensionless quantity.
   (b) Analyze dimensions of equations.
   (c) Compute exponents if we assume joint proportionality.
   (d) Construct dimensionless combinations.
   (e) Nondimensionalize an ODE.
   (f) The Buckingham π Theorem
   (g) Use the Buckingham π Theorem and the Implicit Function Theorem to construct a relationship.

6. Statistics:

   NOTE: I will give you the definitions for the variance, covariance and correlation

1. Definitions of mean, variance, covariance, correlation.

2. If $y = mx + b$, how is the (variance, covariance, correlation) of set y related to the data in set x. (Also see study questions).

3. The line of best fit.

Study Questions

Also see the Dimensional Analysis Exercises and Solutions

1. What does it mean to say that something (e.g., a differential equation) is "dimensionless"?

2. For the line of best fit, what does "best" mean?

3. Explain the difference between: Deterministic vs Stochastic; Continuous vs Discrete; Empirical vs Analytic.

4. Suppose that $x_{k+1} = Ax_k$ for some real number A.

   If we begin with some initial value, x_0, write x_k in terms of A and x_0 (This is called a closed form for the recurrence relation). Is there a fixed point if $|A| < 1$? If $|A| > 1$?

   When will $\sum_{k=0}^{N} x_k$ have a limit as $N \to \infty$? In that case, what is the (infinite) sum?

5. Suppose that some quantity of interest involves the following variables:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

   Find a complete set of dimensionless products.

   Supplementary question: In this problem, we will have some free variables- are we free to choose any two of them to be free?

6. (Use a calculator) We hypothesize that $y \propto z^{1/2}$. Does the following data support this? If so, find the constant of proportionality:

y	6.4	9.1	11.1	12.8	14.3
z	3	6	9	12	15

7. If A is proportional to the square of B, how would we linearize the relationship? Same question, but $A = ke^B$?

8. Prove the two properties for proportions: (i) If $x \propto y$, then $y \propto x$ (ii) If $x \propto y$, $y \propto z$, then $x \propto z$

9. Suppose that we assume that Supply and Demand curves are linear. Here is some hypothetical data (these are relative quantities, so we can have negative prices and quantities). Fill in the missing numbers.

Quantity	Price
3	2
-1	2
-1	14/3
5/3	14/3
?	?
?	?

   Find the equilibrium state- Will the market reach equilibrium?

10. Assume that heat loss is proportional to surface area. How much more heat loss will a 12 inch cube have than a 6 inch cube?

11. Describe Newton's Method graphically and give the formula.

12. For spheres and geometrically similar boxes, show that surface area, S, is proportional to volume V to the 2/3 power. Hint: For the boxes, you might consider a "template box" with measures $1, w, h$. These measures are constant for the whole class, so any particular member is constructed by scaling those. You might call the scaling value k.

13. Suppose that gymnasts are geometrically similar with a characteristic scaling k. First, write each of the following assumptions as mathematical statements:

    (a) Ability is proportional to strength, and inversely proportional to weight.

    (b) Strength is proportional to muscle area.

    Next, how should muscle area scale with k? (Hint: Think of muscle area as proportional to surface area) How should weight scale with k? Put the 4 statements together to show that $A \propto \frac{1}{k}$.

14. Find the volume flow rate of blood flowing in an artery, $\frac{dV}{dt}$, as a function of the pressure drop per unit length, P, the radius r, the density ρ and the viscosity μ.

    (a) Here is a list of dimensions. Fill in the blanks:

Variable	dV/dt	P	r	ρ	μ
Dimension		$M/(LT^2)$	L	M/L^3	$M/(LT)$

    (b) Now perform dimensional analysis and apply Buckingham's π Theorem.

15. The power P delivered to a pump depends on the specific weight w of the fluid pumped, the height h to which the fluid is pumped, and the fluid flow rate q. Use dimensional analysis to determine an equation for power. Below are listed the dimensions:

Variable	P	w	h	q
Dimension	ML^2T^{-3}	$ML^{-2}T^{-2}$	L	L^3T^{-1}

16. What is a "fixed point" for a difference equation of the form $x_{k+1} = f(x_k)$? Find the fixed point(s) if $x_{k+1} = kx_k(1 - x_k)$, k is a real number.

17. State the Buckingham π Theorem.

18. What is the justification for saying that: "If $f(x, y) = 0$, then we can solve for y as a function of x: $y = g(x)$". Apply this to the equation for a unit circle. Using the unit circle, can I always solve for y in an interval about any fixed x^* in the domain?

19. Draw a scatter plot with several data points and draw your estimate of the line of best fit. Recall that we defined our data as (x_i, y_i), with $\hat{y}_i = mx_i + b$ and $y_i = \hat{y}_i + \epsilon_i$. On the plot, show a sample of the following quantities:

    (a) The horizontal line at $\hat{y}$

    (b) $y_i - \overline{y}$ (This is used in computing total variation).

    (c) $y_i - \hat{y}_i$ (This is called the variation that is NOT explained by the model)

    (d) $\hat{y}_i - \overline{y}$ (This is called the variation that IS explained by the model)

20. The crucial element of our modeling of biological systems came from our relationship of surface area to mass, $S \propto M^{2/3}$. How did we arrive at this (Hint: Look back at your answers to question 12).

21. Is it possible that within any single group, we can have an affine[1] relationship with a slope of $-1/3$, but when we put the groups together, we get a relationship with slope $-1/4$? If this is true, should our model have used $-1/3$ or $-1/4$? Discuss.

[1] Affine refers to $y = mx + b$ rather than $y = mx$, which is linear.

22. To understand how branchings scale, let us consider the following model. Take an interval, $[0, 1]$. We'll say that the first "branching" cuts the interval into two pieces, each of equal length. The second branching cuts each of those into two pieces of equal length, and so on. The following table may help.

    (a) Fill in the last entry:

Branching	Number of Intervals	Width of each
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
$\vdots$	$\vdots$	$\vdots$
k		

    (b) Suppose we require that the last branch cover a set width, say $\frac{1}{2^{10}}$. How many branchings k are required?

    (c) Do the same problem, except take the initial interval to have length S.

    (d) Using your previous answer, if length S is multiplied by $2^7 = 128$, how much larger is k?

    (e) If length S is multiplied by c, how much larger is k?

23. We said that, if we have two data sets with n points, x and y, and if they both have zero mean, then the error function (sum of squares) is a function of one variable (model: $y = mx$). Write down the error function and find the slope that minimizes the error.

24. Defining $y_i = mx_i + b + \epsilon_i$, take the mean of both sides to find a formula for b in terms of $m, \overline{x}, \overline{y}$. You may assume that the residuals have zero mean.

25. Show, by using the definition of the covariance, that if we assume $y_i = mx_i + b$, then $S_{xy}^2 = mS_x^2$. Did we need to assume a perfect linear fit for this formula to work?

26. Show, by using the definition of the correlation, that if we assume $y_i = mx_i + b$, then $r_{xy} = \pm 1$.

27. Can the variance produce any real number? Can the covariance? (Give algebraic reasons)

28. Show, using dot products and norms, that the correlation must be a number between ± 1.

29. Suppose our data produces the following summary statistics:

    (a) $S_x^2 = 3$

    (b) $S_{xy}^2 = -7$

    (c) $\overline{x} = 2$

    (d) $\overline{y} = 3$

    Find the line of best best.