Details: Targeting with Air Resistance

In this case, we assume that the frictional force is proportional to the velocity, $F_{\text{friction}} = -kv$. This changes the differential equation for velocity to:

$$\frac{dv}{dt} = a(t) - g - kv(t)$$

Assuming $a(t)$ is constant (we'll change that to piecewise constant shortly), then we have that:

$$\frac{dv}{dt} = (a - g) - kv = c - kv$$

We make this slight change for our convenience in integration. This is called a separable differential equation, and we can solve it by the following method:

$$\frac{1}{c - kv}\, dv = \, dt \Rightarrow \int \frac{1}{c - kv}\, dv = \int \, dt$$

For the right hand integral, let $u = c - kv$ so that $-\frac{1}{k}\, du = dv$. Then

$$-\frac{1}{k}\int \frac{du}{u} = t + C_2 \Rightarrow -\frac{1}{k}\ln|u| = t + C_2 \Rightarrow \ln|c - kv| = -kt + C_3$$

so that:

$$c - kv = \mathrm{e}^{-kt + C_3} = A\mathrm{e}^{-kt}$$

Now we replace c by $a - g$, and we get that:

$$v(t) = -\frac{1}{k}\left(A\mathrm{e}^{-kt} - (a - g)\right)$$

We solve for A by letting $v(0) = v_0$ (Note: The initial velocity in the original problem was 0, but we're going to generalize the equation we get shortly).

$$v_0 = -\frac{1}{k}\left(A - (a - g)\right) \Rightarrow A = (a - g) - kv_0$$

We can now rewrite the velocity equation:

$$v(t) = \frac{a - g}{k}\left(1 - \mathrm{e}^{-kt}\right) + v_0\mathrm{e}^{-kt} \qquad (1)$$

SIDE REMARK: What happens to the velocity equation as $k \to 0$? Using l'Hospital's rule,

$$\lim_{k\to 0}(a-g)\frac{1-\mathrm{e}^{-kt}}{k} + v_0\mathrm{e}^{-kt} = \lim_{k\to 0}(a-g)\frac{1-\mathrm{e}^{-kt}}{k} + \lim_{k\to 0} v_0\mathrm{e}^{-kt} =$$

$$\lim_{k\to 0}(a-g)\frac{t\mathrm{e}^{-kt}}{1} + v_0 = (a-g)t + v_0$$

Which is exactly what we had earlier.

Now, we have our "generic" velocity in Equation 1, we want to write it so that it is valid on the interval $t \in [t_{i-1}, t_i]$:

$$v(t) = \frac{a_i - g}{k}\left(1 - \mathrm{e}^{-k(t-t_{i-1})}\right) + v_{i-1}\mathrm{e}^{-k(t-t_{i-1})}$$

(Note that at $v(t_{i-1}) = v_{i-1}$) Taking $t = t_i$ and letting $\tau = t_i - t_{i-1}$ as we did earlier, we get the difference equation:

$$v_i = \frac{1-\mathrm{e}^{-k\tau}}{k}(a_i - g) + v_{i-1}\mathrm{e}^{-k\tau}$$

Looking at this equation, you should see that k, τ are now constants. With this in mind, we might rewrite this equation, using:

$$p = \frac{1-q}{k}, \quad q = \mathrm{e}^{-k\tau}$$

so that:

$$v_i = pa_i - pg + qv_{i-1}$$

so that:

$$\begin{aligned}
v_1 &= pa_1 - pg \\
v_2 &= pa_2 - pg + q(pa_1 - pg) \\
&= p(a_2 + qa_1) - pg(1+q) \\
v_3 &= pa_3 - pg + q(p(a_2 + qa_1) - pg(1+q)) \\
&= p(a_3 + qa_2 + q^2 a_1) - pg(1 + q + q^2) \\
&\vdots \\
v_k &= p(a_k + qa_{k-1} + \ldots + q^{k-1}a_1) - pg(1 + q + \ldots + q^{k-1})
\end{aligned}$$

From which we get:

$$v_k = p\sum_{j=1}^{k} q^{k-j}a_j - pg\frac{1-q^k}{1-q}$$

We'll now derive the position equations in a similar manner. From our generic velocity equation, we integrate to get the general position equation (the one that give y_0 at time 0):

$$y(t) = \int \frac{a-g}{k}\left(1 - \mathrm{e}^{-kt}\right) + v_0\mathrm{e}^{-kt}\,dt$$

so that:

$$y(t) = \frac{a-g}{k}\left(t + \frac{1}{k}\mathrm{e}^{-kt}\right) - \frac{1}{k}v_0\mathrm{e}^{-kt} + C$$

Put in $t = 0$ and let $y(0) = y_0$, and solve for C. Doing this, our final generic formula for position is:

$$y(t) = \frac{a-g}{k}\left(t + \frac{1}{k}\mathrm{e}^{-kt}\right) - \frac{1}{k}v_0\mathrm{e}^{-kt} + y_0 - \frac{a-g}{k^2} + \frac{v_0}{k}$$

or, regrouping terms:

$$y(t) = y_0 + \frac{a-g}{k}t + \left(\frac{v_0}{k} - \frac{a-g}{k^2}\right)\left(1 - \mathrm{e}^{-kt}\right)$$

To make position valid on the time interval $[t_{i-1}, t_i]$,

$$y(t) = y_{i-1} + \frac{a_i-g}{k}(t - t_{i-1}) + \left(\frac{v_{i-1}}{k} - \frac{a_i-g}{k^2}\right)\left(1 - \mathrm{e}^{-k(t-t_{i-1})}\right)$$

so that at $t = t_i$,

$$y_i = y_{i-1} + \frac{a_i-g}{k}\tau + \left(\frac{v_{i-1}}{k} - \frac{a_i-g}{k^2}\right)\left(1 - \mathrm{e}^{-k\tau}\right)$$

Using substitutions as before, we can rewrite this:

$$p = \frac{1-q}{k}, \quad q = \mathrm{e}^{-k\tau}, \quad r = \frac{\tau - p}{k}$$

so that:

$$y_i = ra_i - rg + pv_{i-1} + y_{i-1}$$

We would like to rewrite this in closed form. Let us compute values of y_i to get the pattern:

$$\begin{aligned}
y_1 &= ra_1 - rg \\
y_2 &= ra_2 - rg + pv_1 + ra_1 - rg \\
&= r(a_2 + a_1 - 2g) + pv_1 \\
y_3 &= ra_3 - rg + pv_2 + r(a_2 + a_1 - 2g) + pv_1 \\
&= r(a_3 + a_2 + a_1 - 3g) + p(v_1 + v_2) \\
&\vdots \\
y_k &= r(a_k + a_{k-1} + \ldots + a_1 - kg) + p(v_1 + v_2 + \ldots + v_{k-1})
\end{aligned}$$

Now let us consider the portion $\sum_{j=1}^{k-1} v_j$. From our previous computation,

$$v_j = p\sum_{i=1}^{j} q^{j-i}a_i - pg\left(\frac{1-q^j}{1-q}\right)$$

so that

$$\sum_{j=1}^{k-1} v_j = \sum_{j=1}^{k-1}\left(p\sum_{i=1}^{j} q^{j-i}a_i - pg\left(\frac{1-q^j}{1-q}\right)\right) =$$

$$\sum_{j=1}^{k-1}\left(p\sum_{i=1}^{j} q^{j-i}a_i\right) - \sum_{j=1}^{k-1} pg\left(\frac{1-q^j}{1-q}\right)$$

The sum on the right can be computed directly:

$$\frac{pg}{1-q}\sum_{j=1}^{k-1}(1-q^j) = \frac{pg}{1-q}\cdot\left((k-1) - \frac{q-q^k}{1-q}\right) =$$

$$pg\frac{(k-1)(1-q)-q+q^k}{(1-q)^2} = pg\frac{k-1-kq+q^k}{(1-q)^2} \tag{2}$$

To compute the sum on the left, we'll switch the order of the sum. To accomplish this more easily, we make an array of the elements in the sum.

	$j=1$	$j=2$	$j=3$	$j=4$	$\dots$	$j=k-1$	
$i=1$	a_1	qa_1	q^2a_1	q^3a_1	$\dots$	$q^{k-2}a_1$	$a_1\sum_{j=0}^{k-2} q^j$
$i=2$		a_2	qa_2	q^2a_2	$\dots$	$q^{k-3}a_2$	$a_2\sum_{j=0}^{k-3} q^j$
$i=3$			a_3	qa_3	$\dots$	$q^{k-4}a_3$	$a_3\sum_{j=0}^{k-4} q^j$
$i=4$				a_4	$\dots$	$q^{k-5}a_4$	$a_4\sum_{j=0}^{k-5} q^j$
$\vdots$						$\vdots$	
$i=k-1$						a_{k-1}	$a_{k-1}\sum_{j=0}^{0} q^j$

Now we see that:

$$\sum_{j=1}^{k-1}\left(p\sum_{i=1}^{j} q^{j-i}a_i\right) = p\sum_{i=1}^{k-1}\left[a_i\left(\sum_{j=0}^{k-1-i} q^j\right)\right] = p\sum_{i=1}^{k-1}\left(\frac{1-q^{k-i}}{1-q}\right)a_i \tag{3}$$

Put Equations 3 and 2 together, and we get:

$$\sum_{j=1}^{k-1} v_j = p\sum_{j=1}^{k-1}\left(\frac{1-q^{k-j}}{1-q}\right)a_j - pg\frac{k-1-kq+q^k}{(1-q)^2} \tag{4}$$

In our sum, taking the index j to k rather than $k-1$ has no effect, since if $j = k$, the term is 0. Therefore, we can rewrite this as:

$$\sum_{j=1}^{k-1} v_j = p \sum_{j=1}^{k} \left(\frac{1 - q^{k-j}}{1 - q} \right) a_j - pg \frac{k - 1 - kq + q^k}{(1 - q)^2} \tag{5}$$

Going back to position, we had:

$$y_k = \sum_{j=1}^{k} r a_j - rkg + p \sum_{j=1}^{k-1} v_j$$

and making the substitution from Equation 5,

$$y_k = \sum_{j=1}^{k} r a_j - rkg + p^2 \sum_{j=1}^{k} \left(\frac{1 - q^{k-j}}{1 - q} \right) a_j - p^2 g \frac{k - 1 - kq + q^k}{(1 - q)^2}$$

From which we get our final form for position:

$$y_k = \sum_{j=1}^{k} \left(r + \frac{p^2 (1 - q^{k-j})}{1 - q} \right) a_j - \frac{g p^2 (k - 1 - kq + q^k)}{(1 - q)^2} - kgr$$