Details: Targeting with Air Resistance

In this case, we assume that the frictional force is proportional to the velocity, \( F_{\text{friction}} = -kv \). This changes the differential equation for velocity to:

\[
\frac{dv}{dt} = a(t) - g - kv(t)
\]

Assuming \( a(t) \) is constant (we’ll change that to piecewise constant shortly), then we have that:

\[
\frac{dv}{dt} = (a - g) - kv = c - kv
\]

We make this slight change for our convenience in integration. This is called a separable differential equation, and we can solve it by the following method:

\[
\int \frac{1}{c - kv} dv = \int dt \quad \Rightarrow \quad \int \frac{1}{c - kv} dv = \int dt
\]

For the right hand integral, let \( u = c - kv \) so that \(-\frac{1}{k} du = dv\). Then

\[
-\frac{1}{k} \int \frac{du}{u} = t + C_2 \quad \Rightarrow \quad -\frac{1}{k} \ln|u| = t + C_2 \quad \Rightarrow \quad \ln|c - kv| = -kt + C_3
\]

so that:

\[
c - kv = e^{-kt+C_3} = Ae^{-kt}
\]

Now we replace \( c \) by \( a - g \), and we get that:

\[
v(t) = -\frac{1}{k} \left(Ae^{-kt} - (a - g)\right)
\]

We solve for \( A \) by letting \( v(0) = v_0 \) (Note: The initial velocity in the original problem was 0, but we’re going to generalize the equation we get shortly).

\[
v_0 = -\frac{1}{k} (A - (a - g)) \quad \Rightarrow \quad A = (a - g) - kv_0
\]

We can now rewrite the velocity equation:

\[
v(t) = \frac{a - g}{k} \left(1 - e^{-kt}\right) + v_0e^{-kt}
\]

(1)
SIDE REMARK: What happens to the velocity equation as $k \to 0$? Using l’Hospital’s rule,

$$\lim_{k \to 0} (a - g) \frac{1 - e^{-kt}}{k} + v_0 e^{-kt} = \lim_{k \to 0} (a - g) \frac{1 - e^{-kt}}{k} + \lim_{k \to 0} v_0 e^{-kt} =$$

$$\lim_{k \to 0} (a - g) \frac{1 - e^{-kt}}{kt} + v_0 = (a - g)t + v_0$$

Which is exactly what we had earlier.

Now, we have our “generic” velocity in Equation 1, we want to write it so that it is valid on the interval $t \in [t_{i-1}, t_i]$:

$$v(t) = \frac{a_i - g}{k} \left(1 - e^{-k(t-t_{i-1})}\right) + v_{i-1} e^{-k(t-t_{i-1})}$$

(Note that at $v(t_{i-1}) = v_{i-1}$) Taking $t = t_i$ and letting $\tau = t_i - t_{i-1}$ as we did earlier, we get the difference equation:

$$v_i = \frac{1 - e^{-k\tau}}{k} (a_i - g) + v_{i-1} e^{-k\tau}$$

Looking at this equation, you should see that $k, \tau$ are now constants. With this in mind, we might rewrite this equation, using:

$$p = \frac{1 - q}{k}, \quad q = e^{-k\tau}$$

so that:

$$v_i = pa_i - pg + q v_{i-1}$$

so that:

$$v_1 = pa_1 - pg$$
$$v_2 = pa_2 - pg + q(pa_1 - pg)$$
$$= p(a_2 + qa_1) - pg(1 + q)$$
$$v_3 = pa_3 - pg + q(p(a_2 + qa_1) - pg(1 + q))$$
$$= p(a_3 + qa_2 + q^2a_1) - pg(1 + q + q^2)$$
$$\vdots$$
$$v_k = p(a_k + qa_{k-1} + \ldots + q^{k-1}a_1) - pg(1 + q + \ldots + q^{k-1})$$

From which we get:

$$v_k = p \sum_{j=1}^{k} q^{k-j}a_j - pg \frac{1 - q^k}{1 - q}$$

2
We’ll now derive the position equations in a similar manner. From our
generic velocity equation, we integrate to get the general position equation
(the one that gives
\( y_0 \) at time 0):

\[
y(t) = \int \frac{a - g}{k} \left( 1 - e^{-kt} \right) + v_0 e^{-kt} \, dt
\]

so that:

\[
y(t) = \frac{a - g}{k} \left( t + \frac{1}{k} e^{-kt} \right) - \frac{1}{k} v_0 e^{-kt} + C
\]

Put in \( t = 0 \) and let \( y(0) = y_0 \), and solve for \( C \). Doing this, our final generic
formula for position is:

\[
y(t) = \frac{a - g}{k} \left( t + \frac{1}{k} e^{-kt} \right) - \frac{1}{k} \frac{a - g}{k^2} + \frac{v_0}{k}
\]
or, regrouping terms:

\[
y(t) = y_0 + \frac{a - g}{k} t + \left( \frac{v_0}{k} - \frac{a - g}{k^2} \right) \left( 1 - e^{-kt} \right)
\]

To make position valid on the time interval \([t_{i-1}, t_i]\),

\[
y(t) = y_{i-1} + \frac{a_i - g}{k} (t - t_{i-1}) + \left( \frac{v_{i-1}}{k} - \frac{a_i - g}{k^2} \right) \left( 1 - e^{-k(t-t_{i-1})} \right)
\]
so that at \( t = t_i \),

\[
y_i = y_{i-1} + \frac{a_i - g}{k} \tau + \left( \frac{v_{i-1}}{k} - \frac{a_i - g}{k^2} \right) \left( 1 - e^{-k\tau} \right)
\]

Using substitutions as before, we can rewrite this:

\[
p = \frac{1 - q}{k}, \quad q = e^{-k\tau}, \quad r = \frac{\tau - p}{k}
\]

so that:

\[
y_i = ra_i - rg + pv_{i-1} + y_{i-1}
\]

We would like to rewrite this in closed form. Let us compute values of \( y_i \) to
get the pattern:

\[
y_1 = ra_1 - rg
y_2 = ra_2 - rg + pv_1 + ra_1 - rg
  = r(a_2 + a_1 - 2g) + pv_1
y_3 = ra_3 - rg + pv_2 + r(a_2 + a_1 - 2g) + pv_1
  = r(a_3 + a_2 + a_1 - 3g) + p(v_1 + v_2)
\vdots
y_k = r(a_k + a_{k-1} + \ldots + a_1 - kg) + p(v_1 + v_2 + \ldots + v_{k-1})
\]
Now let us consider the portion $\sum_{j=1}^{k-1} v_j$. From our previous computation,

$$v_j = p \sum_{i=1}^{j} q^{j-i} a_i - pg \left( \frac{1-q^j}{1-q} \right)$$

so that

$$\sum_{j=1}^{k-1} v_j = \sum_{j=1}^{k-1} \left( p \sum_{i=1}^{j} q^{j-i} a_i - pg \left( \frac{1-q^j}{1-q} \right) \right) = \sum_{j=1}^{k-1} \left( p \sum_{i=1}^{j} q^{j-i} a_i \right) - \sum_{j=1}^{k-1} pg \left( \frac{1-q^j}{1-q} \right)$$

The sum on the right can be computed directly:

$$pg \sum_{j=1}^{k-1} (1-q^j) = pg \sum_{j=1}^{k-1} \left( (k-1) - q - q^k \right) = pg \frac{(k-1)(1-q) - q + q^k}{(1-q)^2}$$

(2)

To compute the sum on the left, we’ll switch the order of the sum. To accomplish this more easily, we make an array of the elements in the sum.

<table>
<thead>
<tr>
<th>$i = 1$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>...</th>
<th>$j = k-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$qa_1$</td>
<td>$q^2a_1$</td>
<td>$q^3a_1$</td>
<td>$q^{k-2}a_1$</td>
<td>...</td>
<td>$a_{k-1}$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$qa_2$</td>
<td>$q^2a_2$</td>
<td>$q^3a_2$</td>
<td>$q^{k-3}a_2$</td>
<td>...</td>
<td>$a_{k-1}$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$qa_3$</td>
<td>...</td>
<td>$q^{k-4}a_3$</td>
<td>...</td>
<td>$a_{k-1}$</td>
<td></td>
</tr>
<tr>
<td>$a_4$</td>
<td>...</td>
<td>$q^{k-5}a_4$</td>
<td>...</td>
<td>$a_{k-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>$a_{k-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{k-1}$</td>
<td>$q^{k-1}a_{k-1}$</td>
<td>...</td>
<td>...</td>
<td>$a_{k-1}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now we see that:

$$\sum_{j=1}^{k-1} \left( p \sum_{i=1}^{j} q^{j-i} a_i \right) = p \sum_{i=1}^{k-1} \left[ a_i \left( \sum_{j=0}^{k-1-i} q^j \right) \right] = p \sum_{i=1}^{k-1} \left( \frac{1-q^{k-i}}{1-q} \right) a_i$$

(3)

Put Equations 3 and 2 together, and we get:

$$\sum_{j=1}^{k-1} v_j = p \sum_{j=1}^{k-1} \left( \frac{1-q^{k-j}}{1-q} \right) a_j - pg \frac{k-1-kq+q^k}{(1-q)^2}$$

(4)
In our sum, taking the index $j$ to $k$ rather than $k - 1$ has no effect, since if $j = k$, the term is 0. Therefore, we can rewrite this as:

$$
\sum_{j=1}^{k-1} v_j = p \sum_{j=1}^{k} \left( \frac{1 - q^{k-j}}{1 - q} \right) a_j - pg \frac{k - 1 - kq + q^k}{(1 - q)^2}
$$

(5)

Going back to position, we had:

$$
y_k = \sum_{j=1}^{k} ra_j - rkg + p \sum_{j=1}^{k-1} v_j
$$

and making the substitution from Equation 5,

$$
y_k = \sum_{j=1}^{k} ra_j - rkg + p^2 \sum_{j=1}^{k} \left( \frac{1 - q^{k-j}}{1 - q} \right) a_j - p^2 g \frac{k - 1 - kq + q^k}{(1 - q)^2}
$$

From which we get our final form for position:

$$
y_k = \sum_{j=1}^{k} \left( r + \frac{p^2(1 - q^{k-j})}{1 - q} \right) a_j - \frac{gp^2(k - 1 - kq + q^k)}{(1 - q)^2} - kgr
$$