

1 Dimensional Analysis Notes

1.1 Introduction

Dimensional analysis is the analysis of a relationship by considering its units of measure. For example, it might be meaningless to construct an equation like:

$$M = T$$

where M is measured in grams and T is measured in time. We will call such an equation *dimensionally inconsistent* or *dimensionally non-homogeneous*. Therefore, an equation that has the correct dimensions will be called *dimensionally homogeneous* (or, dimensionally consistent).

Definition: We will use square brackets to denote the *measure* of a quantity. The measures that are commonly used: M for mass, L for length, and T for time.

Example: What are the units of measure for Force in:

$$F = ma$$

On the right side of the equation, $[ma] = [m][a] = M \cdot \frac{L}{T^2}$, so Force must have units of measure as: MLT^{-2} .

Some guiding principles in dimensional analysis:

1. The measure of a product is the product of the measures. We used this in our example, where $[ma] = [m][a]$.
2. Similarly, we can exponentiate dimensions. For example, $[\sqrt{m}] = \sqrt{[m]} = M^{1/2}$.
3. We can only add terms that have the same units of measure. For example, we cannot add meters to minutes.
4. Physical relationships must be dimensionally consistent (for example, the laws of physics).

Here are some common measures used in physics:

1. Velocity: $[v] = L/T$
2. Volume: $[V] = L^3$
3. Acceleration: $[a] = L/T^2$
4. Density (mass/vol): M/L^3

Some quantities do not have “dimensions”- they are called *dimensionless quantities*. For example, angle measures are dimensionless- why? Consider a circle of radius r , where an arclength s has been identified. Then the angle θ is defined as:

$$\theta = \frac{s}{r}$$

so $[\theta] = L/L = 1$. That is, for a dimensionless quantity x , $[x] = 1$. Some other dimensionless quantities:

- all trigonometric functions
- exponential functions
- logarithms
- quantities which are simply counted, such as the number of people in the room
- plain old numbers (like 2, π , etc.)

Example: What is the dimension of ω in $\sin(\omega t)$ (where t is time)? The dimension is $[\omega] = \frac{1}{T}$ since the sine is dimensionless.

Example: Show that the following equation is dimensionally inconsistent, if v is velocity, x is position, t is time:

$$v^2 = t^2 + \frac{x}{t}$$

Exercise: The following is *Poiseuille's equation* which describes the flow of fluid through a cylindrical tube. The constants are: P is pressure difference¹, $[P] = ML^{-1}T^{-2}$, r is radius, η is viscosity of the fluid, $[\eta] = ML^{-1}T^{-1}$, and l is the length of the tube:

$$\text{flow rate} = \frac{\pi P r^4}{8 \eta l}$$

Here is another version of the same equation, where ρ is the density of the fluid, $[\rho] = ML^{-3}$,

$$\text{flow rate} = \frac{\pi \rho P r^4}{8 \eta l}$$

Use dimensional analysis to determine the dimension(s) of the flow rate- Are these equations the same?

1.2 Dimensional Analysis for Modeling

Many times, we can extract relationships between things by analyzing the associated dimensions. The simplest way to describe this process may be through an example. Here, we are modeling the drag force on an airplane, F_D . Since this is a force, $[F_D] = \frac{ML}{T^2}$. What quantities should F_D depend on?

¹Measured, for example, in lb/in²

We will assume that F_D should be jointly proportional to a cross-sectional area, A , the velocity of the plane, v , and the density of the air, ρ (where $[\rho] = ML^{-3}$). What should the model be? We begin by setting up the desired model:

$$F_D = kA^a v^b \rho^c$$

where we now need to determine the values of a, b, c . Assuming the constant k is dimensionless,

$$[F_D] = [k][A]^a [v]^b [\rho]^c \Rightarrow \frac{ML}{T^2} = (L^2)^a \left(\frac{L}{T}\right)^b \left(\frac{M}{L^3}\right)^c$$

from which we get:

$$MLT^{-2} = M^c L^{2a-3c+b} T^{-b}$$

Equating powers so that the equation is dimensionally consistent gives:

$$c = 1, \quad 1 = 2a - 3c + b \quad -b = -2$$

so the solution is: $a = 1, b = 2, c = 1$. For joint proportions where the quantities of interest are known, this technique works very well. We will revisit this topic later, after we've looked at the Buckingham π Theorem.

1.3 Non-dimensionalization of an ODE

We use differential equations to describe how a certain behavior changes over time. That "behavior" could be position or the size of a population, for example.

It is convenient to have the differential equation be free from the specific units used to measure the behavior. For example, suppose I write a differential equation describing velocity, and I assume length is in miles and time is in seconds. I would have to change the differential equation (or change the variables) if I have my measurements in kilometers and hours. But the dimensions I use are confusing the point- the behavior should be qualitatively the same no matter what units I use- and therefore, I should be able to express a differential equation without regards to units. This is what we mean by a differential equation with dimensionless quantities.

Recall that, if $x(t)$ describes position at time t , then $[x] = L$, $x'(t)$ is velocity with $[x'] = LT^{-1}$, and $x''(t)$ is acceleration with $[x''] = LT^{-2}$.

Consider the differential equation:

$$\frac{d^2 x}{dt^2} = \frac{-gR^2}{(x+R)^2}, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = V_0$$

In this example, g is the acceleration due to gravity, R is the radius of the earth, and V_0 is initial velocity.

We will choose new variables so that the resulting differential equation has no dimensions. First, we write down the variables and the dimensions:

$$[x] = L, [t] = T$$

Now the parameters and those dimensions:

$$[g] = \frac{L}{T^2}, \quad [V_0] = \frac{L}{T}, \quad [R] = L$$

Now, we will let y and τ be replacements for x and t . We will have to group x with a set of parameters so that y will become dimensionless. One good choice:

$$y = \frac{x}{R}$$

and similarly, group parameters with t to make τ dimensionless:

$$\tau = \frac{t}{RV_0^{-1}} = \frac{tV_0}{R}$$

Finally, we need the to compute the derivatives of y . In the end, we will want y to depend on τ . For the term d^2x/dt^2 , we have, by substitution:

$$\frac{d^2x}{dt^2} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d}{dt} \left(\frac{d(yR)}{dt} \right) = R \frac{d}{dt} \left(\frac{dy}{dt} \right)$$

But, writing y as dependent on τ ,

$$R \frac{d}{dt} \left(\frac{dy}{dt} \right) = \frac{d}{dt} \left(\frac{dy}{d\tau} \cdot \frac{d\tau}{dt} \right) = \frac{d}{dt} \left(\frac{dy}{d\tau} \cdot \frac{V_0}{R} \right) = V_0 \frac{d}{dt} \left(\frac{dy}{d\tau} \right)$$

Now, y depends on τ , not t , so we'll rewrite this last expression:

$$V_0 \frac{d}{d\tau} \left(\frac{dy}{d\tau} \right) \cdot \frac{d\tau}{dt} = V_0 \frac{d^2y}{d\tau^2} \cdot \frac{V_0}{R} = \frac{V_0^2}{R} \frac{d^2y}{d\tau^2}$$

Now convert the other side of the differential equation:

$$\frac{-gR^2}{(x+R)^2} = \frac{-gR^2}{(yR+R)^2} = \frac{-g}{(y+1)^2}$$

Finally, we put everything together:

$$\left(\frac{V_0^2}{-gR} \right) \frac{d^2y}{d\tau^2} = \frac{1}{(y+1)^2}, \quad y(0) = 0, \quad \frac{dy}{d\tau}(0) = 1$$

Note that the constant, $\frac{V_0^2}{-gR}$, is dimensionless.

1.3.1 EXAMPLE

In this case, consider the model for the angle of a nonlinear pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin(\theta)$$

Now θ and $\sin(\theta)$ are dimensionless already, so we only need a new variable for t . As before, we list the available parameters and try to come up with a substitution that will be dimensionless:

$$[g] = \frac{L}{T^2}, \quad [l] = L$$

and we'll take, for example:

$$\tau = \frac{t}{\sqrt{l/g}} = \sqrt{\frac{g}{l}} t$$

and double-check that τ is dimensionless:

$$[\tau] = \sqrt{\frac{[g]}{[l]}} [t] = \sqrt{\frac{L/T^2}{L}} T = 1$$

Now we want θ to depend on τ rather than t , so we re-do the differential equation using the chain rule:

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt} \left(\frac{d\theta}{dt} \right) = \frac{d}{dt} \left(\frac{d\theta}{d\tau} \frac{d\tau}{dt} \right) = \frac{d}{dt} \left(\frac{d\theta}{d\tau} \sqrt{\frac{g}{l}} \right) = \sqrt{\frac{g}{l}} \cdot \frac{d}{dt} \left(\frac{d\theta}{d\tau} \right)$$

And one more application of the chain rule:

$$\sqrt{\frac{g}{l}} \cdot \frac{d}{dt} \left(\frac{d\theta}{d\tau} \right) = \sqrt{\frac{g}{l}} \cdot \frac{d}{d\tau} \left(\frac{d\theta}{d\tau} \right) \frac{d\tau}{dt} = \sqrt{\frac{g}{l}} \frac{d^2\theta}{d\tau^2} \cdot \sqrt{\frac{g}{l}}$$

Putting it all together,

$$\frac{d^2\theta}{dt^2} = \frac{g}{l} \frac{d^2\theta}{d\tau^2}$$

so that the original differential equation becomes dimensionless:

$$\frac{g}{l} \frac{d^2\theta}{d\tau^2} = -\frac{g}{l} \sin(\theta) \Rightarrow \frac{d^2\theta}{d\tau^2} = -\sin(\theta)$$

This technique works for more general types of analysis. To summarize our procedure for dimensional analysis:

1. List the variables and their dimensions.
2. List the available parameters and their dimensions.
3. Come up with new variables that will convert the originals to a dimensionless form.

It might be surprising that we can do this *without* a differential equation or an algebraic expression. In fact, we can use dimensional analysis to guess at the relationships between variables. Here's an example.

1.4 EXAMPLE: Analyze the period of a pendulum

Our problem is to understand how the period of a pendulum depends on the relevant physical parameters that describe its motion. We will first list these parameters, then we'll try to come up with dimensionless combinations of them.

mass	m	$[m] = M$
length	l	$[l] = L$
gravity	g	$[g] = \frac{L}{T^2}$
angle	θ	$[\theta] = 1$
period	P	T

We'll call the new variables $\pi_1, \pi_2, \pi_3, \dots$. Again, our goal is to find combinations of the parameters for each π_j so that the result is dimensionless.

We can solve this problem by writing a generic term:

$$\pi = m^a l^b g^c \theta^d P^e$$

so that:

$$[\pi] = [m]^a [l]^b [g]^c [\theta]^d [P]^e = M^a \cdot L^b \cdot \frac{L^c}{T^{2c}} \cdot 1^d \cdot T^e = M^a L^{b+c} T^{e-2c}$$

with d free to be anything. For the other exponents,

$$\begin{aligned} a &= 0 \\ b + c &= 0 \Rightarrow b = -c \\ e - 2c &= 0 \Rightarrow e = 2c \end{aligned}$$

So we see that we have a second free variable, c (that is, given c we can compute b, e). We pause from our discussion for a moment to consider what the implications are for the free variables.

1.4.1 What to do with the free variables?

From our analysis, we saw that a generic variable π had the form:

$$\pi = \theta^d \left(\frac{gP^2}{l} \right)^c$$

Since c and d are free, I might formulate two specific dimensionless variables by setting $c = 1, d = 0$ and $c = 0, d = 1$:

$$\pi_1 = \theta, \quad \pi_2 = \frac{gP^2}{l}$$

Question: Is it possible to construct all other possible dimensionless variables by taking a product of these two? Clearly, yes. We know that all dimensionless products have the form:

$$\theta^a \left(\frac{gP^2}{l} \right)^b = \pi_1^a \pi_2^b$$

In this case, we would say that π_1, π_2 form a *complete set* of variables.

What if I had made the unfortunate choice of taking $c = 1, d = 1$ for π_1 and $c = 2, d = 2$ for π_2 ? Could I still get any dimensionless variable by taking a product of π_1 and π_2 ? No. For example,

$$\theta \left(\frac{gP^2}{l} \right)^2 = \pi_1^a \pi_2^b = \left(\theta \cdot \frac{gP^2}{l} \right)^a \left(\theta^2 \cdot \frac{gP^2}{l} \right)^b$$

Equating exponents, you get: $1 = a + 2b, 2 = a + 2b...$ In this case, π_1 and π_2 do NOT form a complete set.

There is something connected to linear algebra here- The desired sets of free variables should consist of linearly independent choices in order to obtain a complete set. A simpler rule of thumb:

To get a complete set of dimensionless variables, if you have:

- One free variable: Set it to 1.
- Two free variables: Corresponds to two combinations for π , obtained by taking the variables as $(1, 0)$ and $(0, 1)$.
- Three free variables: Take them as $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

and so on.

So far, we have analyzed the dimensionless quantities for the period of the pendulum, and we know that the following form a complete set of dimensionless variables:

$$\pi_1 = \theta, \quad \pi_2 = \frac{gP^2}{l}$$

But how are they related? We can relate them by using a theorem- once we've discussed it, we'll return to our pendulum.

1.4.2 The Buckingham π Theorem:

An equation is dimensionally homogeneous if and only if it can be put into the form:

$$f(\pi_1, \pi_2, \dots, \pi_n) = 0$$

where $\{\pi_1, \pi_2, \dots, \pi_n\}$ is a complete set of dimensionless products.

In using this theorem, we will not normally give the exact functional form of f - remember, we are interested in how the dimensions interact. However, once we know what dimensions are involved, we might use actual data to find the form for f .

At this point, you might wonder what use is this theorem? Even though we do not know the actual function f , there is some analysis that we can do.

Going back to our previous example, we had

$$\pi_1 = \theta, \quad \pi_2 = \frac{gP^2}{l}$$

so that the equation governing the period of the pendulum (being dimensionally homogeneous) must be of the form:

$$f(\pi_1, \pi_2) = f\left(\theta, \frac{gP^2}{l}\right) = 0$$

For now, we will assume that we can solve this equation for $\frac{gP^2}{l}$ as a function of θ . In the past, we have made such assumptions when we perform implicit differentiation. We will not formalize when this assumption is accurate here- it is a topic in real analysis called “The Implicit Function Theorem”.

With our assumption, we let h be the unknown function, and write:

$$\frac{gP^2}{l} = h(\theta)$$

or

$$P = \sqrt{\frac{l}{g}h(\theta)} = \sqrt{\frac{l}{g}} \cdot \hat{h}(\theta)$$

This analysis tells us two things rather immediately:

The period of the pendulum does not depend on the mass. It does depend on the square root of the length.

1.4.3 Example: The velocity of a raindrop

(This is taken from Giordano, Weir and Fox) Consider the problem of determining the terminal velocity of a raindrop falling from a motionless cloud. Can we determine a general model using dimensional analysis?

First, list all the parameters on which we think this will depend, and we'll list the dimensions as well:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

where r is the radius of the raindrop, ρ is the density of the air, and μ is the viscosity.

We now need to form all the dimensionless products. A generic variable π will have the form:

$$\pi = v^a r^b g^c \rho^d \mu^e$$

so that:

$$1 = [\pi] = (LT^{-1})^a L^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$$

Equating exponents (to make the quantity dimensionless), we get three equations in five variables:

$$\begin{aligned} d + e &= 0 \\ a + b + c - 3d - e &= 0 \\ -a - 2c - e &= 0 \end{aligned}$$

Where a, d are arbitrary. For π_1 , we'll take $a = 1$ and $d = 0$. For π_2 , take $a = 0, d = 1$. Using these,

$$\pi_1 = vr^{-1/2}g^{-1/2}, \quad \pi_2 = r^{3/2}g^{1/2}\rho\mu^{-1}$$

(EXERCISE: Verify that these quantities are indeed dimensionless!)

From the Buckingham π Theorem,

$$f(\pi_1, \pi_2) = f\left(vr^{-1/2}g^{-1/2}, r^{3/2}g^{1/2}\rho\mu^{-1}\right) = 0$$

Now we assume that this relationship can be solved for v :

$$vr^{-1/2}g^{-1/2} = h(\pi_2)$$

so that

$$v = \sqrt{rg} \cdot h(\pi_2)$$

This example highlights some of the features and drawbacks to dimensional analysis. On the one hand, we can look at the interaction of variables based on the dimensions, but at the end, we still do not have an exact expression relating the variables.

1.4.4 A couple of final thoughts

Consider the special case if there is only one dimensionless product. Then, the Buckingham π theorem states that there is a function f so that $f(\pi) = 0$. Thus, π is a root to f , and we will assume that there is only one such root. This says that π is a constant,

$$\pi = k$$

Suppose we wish to find a relationship between x and y_1, \dots, y_n , and we can choose some free variables. We would normally choose a as one of the free variables, so that x will appear in only one of the dimensionless products (so that we can extract it with the Implicit Function Theorem).

1.4.5 Example: Wind Force on a Van on Highway

We wish to find a relationship between the force of the wind, F on a van driving down the highway. We expect that the other variables effecting force will be: the velocity of the van v , the surface area of the van exposed to the wind, A , and the density of the air, ρ . Thus we have the following variables and dimensions:

Variable	F	v	A	ρ
Dimension	MLT^{-2}	LT^{-1}	L^2	ML^{-3}

Set up the equations from:

$$\left(\frac{ML}{T^2}\right)^a \left(\frac{L}{T}\right)^b L^{2c} \left(\frac{M}{L^3}\right)^d$$

so that:

$$\begin{aligned}a + d &= 0 \\2a + b &= 0 \\a + b + 2c - 3d &= 0\end{aligned}$$

We have three equations and four unknowns, so we choose a as the free variable (because we're going to want to isolate F). Set $a = 1$ and solve for the other variables to get: $b = -2, c = -1, d = -1$.

Thus, we have one dimensionless product,

$$\pi = \frac{F}{v^2 A \rho}$$

so that the Buckingham π Theorem states there is an f so that

$$f\left(\frac{F}{v^2 A \rho}\right) = 0$$

By our assumption, that means that $\pi = k$, so that

$$\frac{F}{v^2 A \rho} = k, \text{ or } F = k v^2 A \rho$$

Note that this is the same equation as we got for the frictional drag force on an airplane, where we assumed proportionality.

2 Exercises for Dimensional Analysis

1. Nondimensionalize the differential equation:

$$\frac{d^2 x}{dt^2} = \frac{-gR^2}{(x+R)^2}, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = V_0$$

In this example, $[x] = L$, g is the acceleration due to gravity, R is the radius of the earth, and V_0 is initial velocity.

Note that this is the same one as in the text- Try to do it without referring back to it!

2. We will re-dimensionalize the pendulum. That is, we start with the nondimensional form,

$$\frac{d^2 \theta}{d\tau^2} = -\sin(\theta)$$

with the substitution:

$$\tau = \frac{t}{\sqrt{l/g}} = \sqrt{\frac{g}{l}} t$$

Compute the differential equation for $\frac{d^2 \theta}{dt^2}$.

3. Find a proportionality relationship using dimensional analysis of centrifugal force F in terms of mass m , velocity v and radius r .
4. In fluid mechanics, the Reynolds number is a dimensionless number involving fluid velocity v , density ρ , viscosity μ and a characteristic length r . Find this dimensionless product of the variables, given the table of dimensions below (Fill in the missing values first):

Variable	v	ρ	μ	r
Dimension		ML^{-3}	$ML^{-1}T^{-1}$	

5. Certain stars, whose light and radial velocities undergo periodic vibrations, are thought to be pulsing. It is hypothesized that the period t of pulsation depends upon the star's radius, r , its mass, m , and the gravitational constant, G .
- (a) Before going into this problem, as a simpler problem, compute the units of force from: $F = ma$.
- (b) Newton's law of gravitation asserts that the attractive force between two bodies is proportional to the product of their masses divided by the distance between them:

$$F = \frac{Gm_1m_2}{r^2}$$

Compute the units of G from this.

- (c) Going back to our star, express t as an appropriate product of powers of m, r , and G .
6. Find the volume flow rate of blood flowing in an artery, $\frac{dV}{dt}$, as a function of the pressure drop per unit length, P , the radius r , the density ρ and the viscosity μ .

Here is a list of dimensions. Fill in the blanks:

Variable	dV/dt	P	r	ρ	μ
Dimension		$M/(LT^2)$	L	M/L^3	$M/(LT)$

Now perform dimensional analysis and apply Buckingham's π Theorem.

7. The power P delivered to a pump depends on the specific weight w of the fluid pumped, the height h to which the fluid is pumped, and the fluid flow rate q . Use dimensional analysis to determine an equation for power. Below are listed the dimensions:

Variable	P	w	h	q
Dimension	ML^2T^{-3}	$ML^{-2}T^{-2}$	L	L^3T^{-1}