Dimensional Analysis Exercise Solutions

1. Nondimensionalize the differential equation:

$$\frac{d^2x}{dt^2} = \frac{-gR^2}{(x+R)^2}, \quad x(0) = 0, \frac{dx}{dt}(0) = V_0$$

   In this example, $[x] = L$, g is the acceleration due to gravity, R is the radius of the earth, and V_0 is initial velocity.

   Note that this is the same one as in the text- Try to do it without referring back to it!

   SOLUTION: See the text.

2. We will re-dimensionalize the pendulum. That is, we start with the nondimensional form,

$$\frac{d^2\theta}{d\tau^2} = -\sin(\theta)$$

   with the substitution:

$$\tau = \frac{t}{\sqrt{l/g}} = \sqrt{\frac{g}{l}}\, t$$

   Compute the differential equation for $\frac{d^2\theta}{dt^2}$.

   SOLUTION:

   We note that $\frac{d\theta}{d\tau} = \frac{d\theta}{dt} \cdot \frac{dt}{d\tau}$. Given our substitution, $\frac{dt}{d\tau} = \sqrt{\frac{l}{g}}$ so that $\frac{d\theta}{d\tau} = \sqrt{\frac{l}{g}} \frac{d\theta}{dt}$. Repeat the process and substitute back in to get a final answer:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta)$$

3. Find a proportionality relationship using dimensional analysis of centrifugal force F in terms of mass m, velocity v and radius r.

   SOLUTION:

   We set: $F = km^a v^b r^c$, where k is dimensionless, and compare dimensions:

$$\frac{ML}{T^2} = M^a \cdot \frac{L^b}{T^b} \cdot L^c$$

   We see that $a = 1, b = 2$ so $c = -1$. This gives:

$$F = \frac{kmv^2}{r}$$

4. In fluid mechanics, the Reynolds number is a dimensionless number involving fluid velocity v, density ρ, viscosity μ and a characteristic length r. Find this dimensionless product of the variables, given the table of dimensions below (Fill in the missing values first):

Variable	v	ρ	μ	r
Dimension		ML^{-3}	$ML^{-1}T^{-1}$	

SOLUTION:

Set up the dimensionless product:

$$\frac{L^a}{T^a} \cdot \frac{M^b}{L^{3b}} \cdot \frac{M^c}{L^c T^c} \cdot L^d$$

from which we get (M, L, T):

$$\begin{aligned} b + c &= 0 \\ a - 3b - c + d &= 0 \\ -a - c &= 0 \end{aligned}$$

Since we have one free variable, we'll set $a = 1$ to get $b = 1, c = -1, d = 1$. In this case, we have that the dimensionless product that represents the Reynolds number is given by:

$$\frac{v\rho r}{\mu}$$

5. Certain stars, whose light and radial velocities undergo periodic vibrations, are thought to be pulsing. It is hypothesized that the period t of pulsation depends upon the star's radius, r, its mass, m, and the gravitational constant, G.

   (a) Before going into this problem, as a simpler problem, compute the units of force from: $F = ma$.
   SOLUTION: $[F] = [m][a] = M \cdot LT^{-2} = \frac{ML}{T^2}$

   (b) Newton's law of gravitation asserts that the attractive force between two bodies is proportional to the product of their masses divided by the distance between them:

   $$F = \frac{Gm_1 m_2}{r^2}$$

   Compute the units of G from this.
   SOLUTION: $\dfrac{ML}{T^2} = [G]\dfrac{M^2}{L^2} \Rightarrow [G] = \dfrac{L^3}{MT^2}$

   (c) Going back to our star, express t as an appropriate product of powers of m, r, and G.

SOLUTION: $t = m^a r^b G^c \Rightarrow T = M^a L^b \frac{L^{3c}}{M^c T^{2c}}$ so:

$$a - c = 0, b + 3c = 0, -2c = 1 \Rightarrow a = -\frac{1}{2}, b = \frac{3}{2}, c = -\frac{1}{2}$$

Now, $t = \dfrac{r^{3/2}}{\sqrt{mG}}$

6. Find the volume flow rate of blood flowing in an artery, $\frac{dV}{dt}$, as a function of the pressure drop per unit length, P, the radius r, the density ρ and the viscosity μ.

   Here is a list of dimensions. Fill in the blanks:

Variable	dV/dt	P	r	ρ	μ
Dimension		$M/(LT^2)$	L	M/L^3	$M/(LT)$

   Now perform dimensional analysis and apply Buckingham's π Theorem.

   SOLUTION: A generic dimensionless variable will have the form:

$$\pi = (dV)^a P^b r^c \rho^d \mu^e$$

   so that, in dimensions,

$$1 = \frac{L^{3a}}{T^a} \cdot \frac{M^b}{L^b T^{2b}} \cdot L^c \cdot \frac{M^d}{L^{3d}} \cdot \frac{M^e}{L^e T^e}$$

   and by equating exponents (L,M, then T):

$$\begin{array}{rrrrrl} 3a & -b & +c & -3d & -e & = 0 \\ & b & & +d & +e & = 0 \\ -a & -2b & & & -e & = 0 \end{array}$$

   We have three equations and five unknowns, so we can choose the two free variables. Since we're going to want to solve for dV/dt, we'll choose a to be one of the free ones. Arbitrarily, we choose e as the second.

   There are several ways of writing the solutions. But notice that, given the choice we made for free variables, we can write:

$$\begin{aligned} -b + c - 3d &= -3a + e \\ b + d &= -e \\ -2b &= a + e \end{aligned}$$

   Which we could leave as is, or we can go ahead and solve, getting:

$$b = -\frac{1}{2}a - \frac{1}{2}e, \quad c = -2a - e, \quad d = \frac{1}{2}a - \frac{1}{2}e$$

   Substituting $a = 2, e = 0$ and $a = 0, e = 2$, we get:

   $a = 2, b = -1, c = -4, d = 1, e = 0$ and $a = 0, b = -1, c = -2, d = -1, e = 2$

Therefore,

$$\pi_1 = \left(\frac{dV}{dt}\right)^2 \frac{\rho}{Pr^4}, \quad \pi_2 = \frac{\mu^2}{Pr^2\rho}$$

By the Buckingham π Theorem, there exists an f so that:

$$f\left(\left(\frac{dV}{dt}\right)^2 \frac{\rho}{Pr^4}, \frac{\mu^2}{Pr^2\rho}\right) = 0$$

By the Implicit Function Theorem, we assume that we can solve this so that there exists a function h:

$$\left(\frac{dV}{dt}\right)^2 \frac{\rho}{Pr^4} = h\left(\frac{\mu^2}{Pr^2\rho}\right)$$

so that:

$$\frac{dV}{dt} = r^2 \cdot \sqrt{\frac{P}{\rho}} \cdot h\left(\frac{\mu^2}{Pr^2\rho}\right)$$

7. The power P delivered to a pump depends on the specific weight w of the fluid pumped, the height h to which the fluid is pumped, and the fluid flow rate q. Use dimensional analysis to determine an equation for power. Below are listed the dimensions:

Variable	P	w	h	q
Dimension	ML^2T^{-3}	$ML^{-2}T^{-2}$	L	L^3T^{-1}

Set up the equations from:

$$\left(\frac{ML^2}{T^3}\right)^a \left(\frac{M}{L^2T^2}\right)^b L^c \left(\frac{L^3}{T}\right)^d$$

From which:

$$\begin{aligned} a+b &= 0 \\ -3a-2b-d &= 0 \\ 2a-2b+c+3d &= 0 \end{aligned}$$

Take a as the free variable, and set it to 1, then $b = c = d = -1$. This gives one dimensionless variable,

$$\pi_1 = \frac{P}{whq}$$

so that the Buckingham π Theorem states that there is a function f so that

$$f\left(\frac{P}{whq}\right) = 0$$

Since $f(\pi) = 0$, π is a constant. Thus,

$$P \propto whq \text{ or } P = kwhq$$