Exam 2 Overview Questions, Math 250

This exam will cover the linear and nonlinear programming portion of the course. Be sure to review your homework problems, and be able to answer the following questions.
Please bring a calculator to the exam. The exam will be "closed book" and "closed notes".

1. True or False, and explain:

   (a) In a nonlinear program, the domain D is convex.

   (b) It is possible that a solution to a linear program is in the middle of an edge to the domain.

   (c) Let the domain D of a nonlinear program be given as $A\mathbf{x} \le \mathbf{b}$. It is possible that a solution is in the middle of an edge of D.

   (d) The following set of points forms a convex set:

$$\left\{ \mathbf{x} \,:\, \begin{bmatrix} 3 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right\}$$

2. Discussion/Short Answer:

   (a) State the "Fundamental Theorem of Linear Programming":

   (b) State the Extreme Value Theorem:

   (c) What is a "linear program"?

   (d) Finish the definition: A set D is convex if:

   (e) Under what conditions may we get no solution to a linear program? Does this change for a nonlinear program?

   (f) If $f : \mathbb{R}^n \to \mathbb{R}^n$, then the linearization of f at $\mathbf{x} = \mathbf{a}$ is:

   (g) In the targeting problem, how did we change the problem so that the objective function, $|a_1| + |a_2| + \ldots + |a_N|$ became linear?

   (h) If $f : \mathbb{R}^n \to \mathbb{R}$, then the linearization of f at $\mathbf{x} = \mathbf{a}$ is:
   Given the same f, the second order Taylor approximation is:

   (i) If we state that, "Given $g(x, y) = k$, it is possible under certain conditions to write this as $y = \hat{g}(x)$", what theorem are we using?

   (j) Comment on the following: If I lived in a house with a convex floor plan, I could see everything in the house from any point within it.

3. Suppose acceleration is some constant, $a = 40$ so that velocity, with air resistance, is given by:

$$\frac{dv}{dt} = 40 - 32 - kv(t) = 8 - kv(t)$$

If $v(0) = 0$ and $y(0) = 0$, what is the height that can be reached at time 20? (NOTE: Set up the equation that you would have to solve in terms of k, and the height h, but don't solve it).

4. (Referring to the previous question) Use Newton's Method in Matlab to approximate the value of k if the desired height is 100. Begin the approximation with $k_0 = 0.8$ and do Newton for 10 iterations.

5. Consider the linear program: $\max 3x + y$ such that $x, y \geq 0$, $x + y \geq 2$, $x - y \leq 2$ and $y \leq 3$.

   (a) Graph the feasible region and label all vertices of the polytope.

   (b) Solve the LP graphically.

   (c) Show that this is the solution by using the vertices.

6. Use Lagrange multipliers to find the maximum of $4x + 6y$ if $x^2 + y^2 = 16$.

7. Find the critical points of $f(x, y) = 3x - x^3 - 2y^2 + y^4$.

8. Find the global maximum of $f(x, y) = 1 + xy - x - y$ on the domain D bounded by the parabola $y = x^2$ and the line $y = 4$.

9. Recall that for a 2×2 matrix, we can compute the inverse directly:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   Use this formula to compute one step of multidimensional Newton's method, if $f(x, y)$ is given below, and the inital guess is $(1, 1)$.

$$f(x, y) = \begin{bmatrix} x^2 + 2y \\ 3xy - y^2 \end{bmatrix}$$

10. Let $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z}$. At the point $(4, 2, 1)$, find the maximum rate of change of f and the direction in which it occurs.

11. Formulate the following problem as an LP. Do not solve it:

    A confectioner manufactures two kinds of candy bars; "ProteinPlus", that has no carbohydrates, and "SugarPlus", with no fat. ProteinPlus sells for a profit of 40 cents per bar, and SugarPlus sells for a profit of 50 cents per bar. The candy is processed in three main operations; blending, cooking and packaging. The following table records the average time in minutes required by each bar for each of the processing operations:

	Blending	Cooking	Packaging
ProteinPlus	1	5	3
SugarPlus	2	4	1

During each production run, the blending equipment is available for a maximum of 12 machine hours, the cooking equipment is available for at most 30 machine hours, and the packaging equipment for no more than 15 hours. If this machine time can be allocated to the making of either type of candy at all times that it is available, determine how many boxes of each kind the confectioner should make in order to realize the maximum profit.

12. (Referring to the previous problem) Solve the LP problem.

13. Fill in the blank with *perpendicular* or *parallel*. Also answer the associated questions.

    (a) Let $f : \mathbb{R} \to \mathbb{R}^2$. The line at $f(2)$ in the direction of $f'(2)$ is ____________ to the curve $f(t)$.

    (b) Let $z = f(x, y)$. The gradient of f at (a, b) is ____________ to a level curve of f. Show your answer is correct, if $z = f(x, y) = x^2 + 3xy$ at the point $(1, 1)$. Which level curve are you on?

    (c) Let $k = f(x, y)$ be a level curve that can be written as $y = g(x)$. Then at a point (a, b), the gradient of f at (a, b) is ____________ to the line $y - b = g'(a)(x - a)$.

    (d) Suppose $z = f(x, y)$ with the domain restriction that $g(x, y) = k$. If the optimum of f (with the restricted domain) is (a, b), then:

    - Does $\nabla f(a, b) = 0$?
    - $\nabla f(a, b)$ is ____________ to $\nabla g(a, b)$

14. Compute the gradient and Hessian of $f(x, y) = 5 - 7y + 8x^2y + 3y^2$.

15. Write the equations necessary to solve the following problem (but do not solve it):

    Suppose there are n points on the unit sphere. Find the positions of the points so that they all have maximal separation from each other. (This is from a physics problem, where the points are electrons, and they repel each other).