Exam 2 Review SOLUTIONS, Math 250

1. True or False, and explain:

   (a) In a nonlinear program, the domain D is convex.

   False- not necessarily. In the nonlinear case, we might have very complex curves and complex domains.

   (b) It is possible that a solution to a linear program is in the middle of an edge to the domain.

   False. We showed in class than a solution to a *linear* program will occur at a vertex along the feasible set.

   (c) Let the domain D of a nonlinear program be given as $A\mathbf{x} \le \mathbf{b}$. It is possible that a solution is in the middle of an edge of D.

   This is possible, and we saw it in one of the examples done in class.

   (d) The following set of points forms a convex set:

$$\left\{ \mathbf{x} : \begin{bmatrix} 3 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right\}$$

   TRUE: Any (nonempty) set of the form $Ax \le b$ forms a convex set.

2. Discussion/Short Answer:

   (a) State the "Fundamental Theorem of Linear Programming":

   Given $f(x) = C^T x$ such that $Ax \le b$, which is nonempty, then there is a maximum and minimum value of f, and it is attained at a vertex of the polytope.

   (b) State the Extreme Value Theorem: Let f be continuous over a closed and bounded domain D. Then f attains a maximum and minimum value on D.

   (c) What is a "linear program"? It is an optimization problem of the form: $\max_x c^T x$ such that $Ax \le b$.

   (d) Finish the definition: A set D is convex if: one takes any two points in D, all the points in between them are also in D.

   (e) Under what conditions may we get no solution to a linear program? Does this change for a nonlinear program?

   It might be that the domain D forms an inconsistent set of equations (there is no feasible set). It might also be that D is unbounded. That's it, since if D is a closed and bounded set, then the linear program must have a solution.

   It doesn't change for a nonlinear program- the given constraint equations might be inconsisten, or the domain restrictions may form an unbounded set.

(f) If $f : \mathbb{R}^n \to \mathbb{R}^n$, then the linearization of f at $\mathbf{x} = \mathbf{a}$ is:

$$L(x) = f(\mathbf{a}) + Jf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

(g) In the targeting problem, how did we change the problem so that the objective function, $|a_1| + |a_2| + \ldots + |a_N|$ became linear?

We created new variables $x_{N+1}, \ldots, x_{2N}$ so that:

$$x_{N+i} = |x_i| \Rightarrow\Rightarrow \left.\begin{array}{ll} x_{N+i} & \geq x_i \\ x_{N+i} & \geq -x_i \end{array}\right\} \text{ for } i = 1, 2, \ldots, N$$

(h) If $f : \mathbb{R}^n \to \mathbb{R}$, then the linearization of f at $\mathbf{x} = \mathbf{a}$ is:

$$L(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

(Note that the last term is a dot product).

Given the same f, the second order Taylor approximation is:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

(i) If we state that, "Given $g(x, y) = k$, it is possible under certain conditions to write this as $y = \hat{g}(x)$", what theorem are we using? The implicit function theorem.

(j) Comment on the following: If I lived in a house with a convex floor plan, I could see everything in the house from any point within it.

True, if you don't have any big furniture. Your sight line should always be unobstructed to the boundary of the house.

3. Suppose acceleration is some constant, $a = 40$ so that velocity, with air resistance, is given by:

$$\frac{dv}{dt} = 40 - 32 - kv(t) = 8 - kv(t)$$

If $v(0) = 0$ and $y(0) = 0$, what is the height that can be reached at time 20? (NOTE: Set up the equation that you would have to solve in terms of k, and the height h, but don't solve it).

We did this one partially in class:

$$\frac{1}{8 - kv}\, dv = dt \Rightarrow \int \frac{1}{8 - kv}\, dv = \int dt$$

so that

$$-\frac{1}{k} \ln|8 - kv| = t + C_1 \Rightarrow 8 - kv = \mathrm{e}^{-kt - kC_1} = A\mathrm{e}^{-kt}$$

or

$$v = \frac{8}{k} - \frac{A}{k}\mathrm{e}^{-kt}, \quad v(0) = 0 \Rightarrow A = 8$$

therefore, we get a final version of $v(t)$:

$$v(t) = \frac{8}{k}\left(1 - \mathrm{e}^{-kt}\right)$$

To get $y(t)$, antidifferentiate v:

$$y(t) = \frac{8}{k}t + \frac{8}{k^2}\mathrm{e}^{-kt} + C$$

To have $y(0) = 0$, we get that $C = -\frac{8}{k^2}$,

$$y(t) = \frac{8}{k}t + \frac{8}{k^2}\left(\mathrm{e}^{-kt} - 1\right)$$

Our final answer would be:

$$h = \frac{160}{k} + \frac{8}{k^2}\left(\mathrm{e}^{-20k} - 1\right)$$

4. (Referring to the previous question) Use Newton's Method in Matlab to approximate the value of k if the desired height is 100. Begin the approximation with $k_0 = 0.8$ and do Newton for 10 iterations.

   (Matlab won't be on the exam, but the solution is approximately 1.5483)

5. Consider the linear program: $\max 3x + y$ such that $x, y \geq 0$, $x + y \geq 2$, $x - y \leq 2$ and $y \leq 3$.

   (a) Graph the feasible region and label all vertices of the polytope. Check your answer on Maple.

   (b) Solve the LP graphically. See the solution below.

   (c) Show that this is the solution by using the vertices.
   The solution is: $x = 5, y = 3$

6. Use Lagrange multipliers to find the maximum of $4x + 6y$ if $x^2 + y^2 = 16$.

   We have the system of equations:

$$\begin{aligned} 4 &= \lambda 2x \\ 6 &= \lambda 2y \\ x^2 + y^2 &= 16 \end{aligned}$$

   As we did in class, we can multiply the first equation by y and the second equation by x to get that $y = \frac{3}{2}x$. Substitute this into the third equation to solve for y.

7. Find the critical points of $f(x, y) = 3x - x^3 - 2y^2 + y^4$.

   We check for where the gradient is zero. In this example,

$$\nabla f = [3 - 3x^2,\ -4y + 4y^3]$$

   Altogether, we have six points:

$$(1, -1), (1, 0), (1, 1), (-1, -1), (-1, 0), (-1, 1)$$

8. Find the global maximum of $f(x, y) = 1 + xy - x - y$ on the domain D bounded by the parabola $y = x^2$ and the line $y = 4$.

   We first check the critical points of f:

$$\nabla f = [y - 1, x - 1]$$

   so that the only critical point is $(1, 1)$, and $f(1, 1) = 0$.

   We now check the boundary in two parts:

   - $y = 4$, $-2 \leq x \leq 2$. Now $f(x) = 1+4x-x-4 = -3+3x$. On the interval $[-2, 2]$, this function has its maximum at $x = 2$. For this point, $f(2, 4) = -3 + 6 = 3$.
   - $y = x^2$, $-2 \leq x \leq 2$. Now $f(x) = 1 + x^3 - x - x^2$, or $f(x) = x^3 - x^2 - x + 1$. To find its maximum, check the critical points and the endpoints:

$$f'(x) = 3x^2 - 2x - 1 \Rightarrow x = 1, -1/3$$

     Now we check the value of f at $x = -2, -1/3, 1, 2$:

$$f(-2) = -9, f(-1/3) = 32/27 \approx 1.19, f(1) = 0, f(2) = 3$$

     so once again the maximum is at $x = 2, y = 4$.

   Overall, we have found the maximum value to be 3, which occurs at $(2, 4)$.

9. Recall that for a 2×2 matrix, we can compute the inverse directly:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   Use this formula to compute one step of multidimensional Newton's method, if $f(x, y)$ is given below, and the inital guess is $(1, 1)$.

$$f(x, y) = \begin{bmatrix} x^2 + 2y \\ 3xy - y^2 \end{bmatrix}$$

In this case, we use the formula:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \left(Jf(\mathbf{x}_k)\right)^{-1} f(\mathbf{x}_k)$$

The Jacobian matrix is given by:

$$Jf(x,y) = \begin{bmatrix} 2x & 2 \\ 3y & 3x-2y \end{bmatrix}, \quad Jf(1,1) = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } f(1,1) = [3,2]^T$$

so that at the point $(1,1)$,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ -1/4 \end{bmatrix}$$

10. Let $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z}$. At the point $(4,2,1)$, find the maximum rate of change of f and the direction in which it occurs.

    The maximum rate of change of f occurs in the direction of the gradient. The rate at which the change occurs is the size (or norm) of the gradient:

$$\nabla f = \left[\frac{1}{y},\ -\frac{x}{y^2} + \frac{1}{z},\ -\frac{y}{z^2}\right]$$

    At the point $(4,2,1)$, we have that this direction is $\left[\frac{1}{2}, 0, -2\right]$ and the rate of change is $\frac{\sqrt{17}}{2}$

11. Formulate the following problem as an LP. Do not solve it:

    A confectioner manufactures two kinds of candy bars; "ProteinPlus", that has no carbohydrates, and "SugarPlus", with no fat. ProteinPlus sells for a profit of 40 cents per bar, and SugarPlus sells for a profit of 50 cents per bar. The candy is processed in three main operations; blending, cooking and packaging. The following table records the average time in minutes required by each bar for each of the processing operations:

	Blending	Cooking	Packaging
ProteinPlus	1	5	3
SugarPlus	2	4	1

    During each production run, the blending equipment is available for a maximum of 12 machine hours, the cooking equipment is available for at most 30 machine hours, and the packaging equipment for no more than 15 hours. If this machine time can be allocated to the making of either type of candy at all times that it is available, determine how many boxes of each kind the confectioner should make in order to realize the maximum profit. See the solution below.

12. (Referring to the previous problem) Solve the LP problem. The solution is given by $P = 120, S = 300$. You can check this and plot the feasible region in Maple:

```
restart;
with(simplex):
CS:=P+2*S<=12*60, 5*P+4*S<=30*60, 3*P+S<=15*60;
maximize(0.4*P+0.5*S,[CS],NONNEGATIVE);
with(plots):
inequal({CS},P=0..800,S=0..800,optionsfeasible=(color=
white), optionsexcluded=(color=gray));
```

13. Fill in the blank with *perpendicular* or *parallel*. Also answer the associated questions.

    (a) Let $f : \mathbb{R} \to \mathbb{R}^2$. The line at $f(2)$ in the direction of $f'(2)$ is parallel (or tangent) to the curve $f(t)$.

    (b) Let $z = f(x, y)$. The gradient of f at (a, b) is perpendicular to a level curve of f. Show your answer is correct, if $z = f(x, y) = x^2 + 3xy$ at the point $(1, 1)$. Which level curve are you on?
    We are on the level curve $f(x, y) = f(1, 1) = 1+3 = 4$. In this particular example, this would mean that we check to see if the gradient, $[2x + 3, 3x]$ at $(1, 1)$ giving a gradient of $[5, 3]$ is perpendicular to the surface $x^2 + 3xy = 4$ at $(1, 1)$. To check this, we need the slope of the tangent line to the curve at $(1, 1)$ which we can get by implicit differentiation:

    $$2x + 3y + 3xy' = 0 \Rightarrow y' = -\frac{2x + 3y}{3x}$$

    At $(1, 1)$, the slope is $-5/3$. A line in the direction of the gradient would have a slope of $3/5$, so they are indeed perpendicular.

    (c) Let $k = f(x, y)$ be a level curve that can be written as $y = g(x)$. Then at a point (a, b), the gradient of f at (a, b) is *perpendicular* to the line $y - b = g'(a)(x - a)$.

    (d) Suppose $z = f(x, y)$ with the domain restriction that $g(x, y) = k$. If the optimum of f (with the restricted domain) is (a, b), then:

    - Does $\nabla f(a, b) = 0$? Not necessarily, although it might happen by chance. Generally, we do not expect f to actually have critical points on the set $g(x, y) = k$.
    - $\nabla f(a, b)$ is *parallel* to $\nabla g(a, b)$

14. Compute the gradient and Hessian of $f(x, y) = 5 - 7y + 8x^2y + 3y^2$.

$$\nabla f = [16xy \quad -7 + 8x^2 + 6y]$$

$$Hf = \begin{bmatrix} 16y & 16x \\ 16x & 6 \end{bmatrix}$$

15. Write the equations necessary to solve the following problem (but do not solve it):

Suppose there are n points on the unit sphere. Find the positions of the points so that they all have maximal separation from each other. (This is from a physics problem, where the points are electrons, and they repel each other).

One way to write this problem: Let $\{(x_i, y_i, z_i)\}_{i=1}^{n}$ be the unknown points on the sphere. Then:

$$\min \sum_{i=1}^{n} \sum_{j=i+1}^{n} (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$$

such that:

$$x_i^2 + y_i^2 + z_i^2 = 1, \quad i = 1 \ldots n$$

You might notice that this does not have a unique solution, as there are an infinite number of ways of placing the n points on a sphere. One way to approach the solution of this problem would be to initialize the n points at random, then update the positions until we get to an approximate solution.

This was a problem I actually ran into when I was working on a method to "triangulate" the surface of a lung.