An Overview: Chapters 1 and 2

1. Introductory Material: Deterministic vs Stochastic, Continuous vs Discrete, Empirical vs Analytic, Newton's Method, Difference Equations.

2. Graphical Analysis

   (a) Definitions: Function iteration, fixed points (equilibria), attracting/repelling equilibria.

   (b) Graphical analysis of $x_{k+1} = f(x_k)$

   (c) Graphical analysis of competing processes (supply/demand or extinction/migration).

   (d) Case Study: Sociology- Group behavior vs. Individual Thresholds

3. Proportions

   (a) Definitions: A proportional to B, A inversely proportional to B, A is jointly proportional to B and C.

   (b) Properties:

      i. $A \propto B \Rightarrow B \propto A$.

      ii. $A \propto B$, $B \propto C \Rightarrow A \propto C$.

4. Scaling

   (a) Definitions: Geometrically similar, Class of geometrically similar objects, Characteristic length, Template for the class, Allometric analysis.

   (b) Linearize scaling relationships for analysis in Matlab.

   (c) Scaling of area, surface area and volume in Rectangles, Boxes, Circles, Spheres.

   (d) Case Study: Biological Scaling

5. Dimensional Analysis

   (a) Definitions: Dimensionally consistent (or dimensionally homogeneous), dimensionless quantity.

   (b) Analyze dimensions of equations.

   (c) Compute exponents if we assume joint proportionality.

   (d) Construct dimensionless combinations.

   (e) Nondimensionalize an ODE.

   (f) The Buckingham π Theorem

   (g) Use the Buckingham π Theorem to construct a relationship.

Study Questions

Also see the Dimensional Analysis Exercises and Solutions

1. What does it mean to say that something (e.g., a differential equation) is "dimensionless"?

2. Explain the difference between: Deterministic vs Stochastic; Continuous vs Discrete; Empirical vs Analytic.

3. Suppose that $x_{k+1} = Ax_k$ for some real number A.

   If we begin with some initial value, x_0, write x_k in terms of A and x_0 (This is called a closed form for the recurrence relation). Is there a fixed point if $|A| < 1$? If $|A| > 1$?

   When will $\sum_{k=0}^{N} x_k$ have a limit as $N \to \infty$? In that case, what is the (infinite) sum?

4. Is it possible to use a "simple" model to produce complicated behavior? Discuss in terms of what you did in our first set of exercises.

5. Suppose that some quantity of interest involves the following variables:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

   Find a complete set of dimensionless products.

   Supplementary question: In this problem, we will have some free variables- are we free to choose any two of them to be free?

6. (Use a calculator) We hypothesize that $y \propto z^{1/2}$. Does the following data support this? If so, find the constant of proportionality:

y	6.4	9.1	11.1	12.8	14.3
z	3	6	9	12	15

7. If A is proportional to the square of B, how would we linearize the relationship? Same question, but $A = ke^B$?

8. Prove the two properties for proportions: (i) If $x \propto y$, then $y \propto x$ (ii) If $x \propto y$, $y \propto z$, then $x \propto z$

9. Suppose that we assume that a Migration and Extinction curves are linear. Here is some data that I've "retrieved from the field"- "0" in the number of species refers to the current number, so negative values are valid. Fill

in the missing numbers.

Number of Species	Rate
3	2
-1	2
-1	14/3
5/3	14/3
?	?
?	?

Find the equilibrium state. Predict what happens, if in a particularly good summer, there is much more food on the island than usual. In particular, is it possible that the equilibrium does not change?

10. Assume that heat loss is proportional to surface area. How much more heat loss will a 12 inch cube have than a 6 inch cube?

    Now consider two irregularly shaped objects, such as submarines. If we can measure heat loss from a 7 foot scale model, how much more heat is lost from a 70 foot sub?

11. Describe Newton's Method graphically and give the formula.

12. For spheres and geometrically similar boxes, show that surface area, S, is proportional to volume V to the 2/3 power. Hint: For the boxes, you might consider a "template box" with measures $1, w, h$. These measures are constant for the whole class, so any particular member is constructed by scaling those.

13. Suppose that gymnasts are geometrically similar with a characteristic scaling k. First, write each of the following assumptions as mathematical statements:

    (a) Ability is proportional to strength, and inversely proportional to weight.

    (b) Strength is proportional to muscle area.

    Next, how should muscle area scale with k? (Hint: Think of muscle area as proportional to surface area) How should weight scale with k? Put the 4 statements together to show that $A \propto \frac{1}{k}$.

14. What is a "fixed point" for a difference equation of the form $x_{k+1} = f(x_k)$? Find the fixed point(s) if $x_{k+1} = kx_k(1 - x_k)$, k is a real number.

15. State the Buckingham π Theorem.

16. What is "allometric" scaling?

17. What is the justification for saying that: "If $f(x, y) = 0$, then we can solve for y as a function of x: $y = g(x)$". Apply this to the equation for a unit circle. Using the unit circle, can I always solve for y in an interval about any fixed x^* in the domain?

18. Sociologists will use “Group norms” in order to talk about group behavior (how might you define a “group norm” mathematically?). (i) What does our Case Study say about how a group may behave with respect to a group norm versus individual thresholds? For example, will a single individual’s decision be a crucial element to a group norm? In our individual threshold model? (ii) How might a group norm be effected by a set of individual thresholds, and vice versa?

19. Come up with a situation (graphically) where a set of individual thresholds produce a model that has three equilibria, one at the origin, one at $(100, 100)$, and the middle equilibrium is repelling.

20. Same question as before, but the middle equilibrium is attracting.

21. The crucial element of our modeling of biological systems came from our relationship of surface area to mass, $S \propto M^{2/3}$. How did we arrive at this?

22. Is it possible that within any single group, we can have an affine[1] relationship with a slope of $-1/3$, but when we put the groups together, we get a relationship with slope $-1/4$? If this is true, should our model have used $-1/3$ or $-1/4$? Discuss.

23. To understand how branchings scale, let us consider the following model. Take an interval, $[0, 1]$. We’ll say that the first “branching” cuts the interval into two pieces, each of equal length. The second branching cuts each of those into two pieces of equal length, and so on. The following table may help. Fill in the last entry:

Branching	Number of Intervals	Width of each
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
$\vdots$	$\vdots$	$\vdots$
k		

    Suppose we require that the last branch cover a set width, say $\frac{1}{2^{10}}$. How many branchings are required?

    Do the same problem, except take the initial interval to have length S.

    Using your previous answer, if length S is multiplied by $2^7 = 128$, how much larger is k?

    If length S is multiplied by c, how much larger is k?

    NOTE: It might be interesting at some point to see how this would work using volumes instead of intervals, with d branchings instead of 2.

[1]Affine refers to $y = mx + b$ rather than $y = mx$, which is linear.