Study Question Solutions

Also see the Dimensional Analysis Exercises and Solutions

1. What does it mean to say that something (e.g., a differential equation) is "dimensionless"?

   A quantity is dimensionless if it does not have a dimension (such as mass, length or time) attached to its unit of measure. To say that a differential equation is dimensionless means much the same thing- no mass, length or time is being measured.

2. Explain the difference between: Deterministic vs Stochastic; Continuous vs Discrete; Empirical vs Analytic.

   Deterministic/Stochastic: Follows from some algebraic formula (like $y = f(x)$), while Stochastic means that we have to use probabilistic descriptions. For example, in a deterministic difference equation, I can compute one exact value for the future state, but in a stochastic model, I must compute a probability of being in a particular future state.

   Continuous/Discrete: Refers to how time is being measured- either continuously (like in a differential equation) or in discrete samples of time (like a difference equation).

   Empirical/Analytic: An empirical model is one that is derived solely from data. An analytic model is one that is derived from the principles that we think govern the object under study. That is, we can explain the data using an empirical model, but it gives no hint as to *why* the data is formed that way.

3. Suppose that $x_{k+1} = Ax_k$ for some real number A.

   If we begin with some initial value, x_0, write x_k in terms of A and x_0 (This is called a closed form for the recurrence relation).

   $$\text{Solution: } x_k = A^k x_0$$

   Is there a fixed point if $|A| < 1$? If $|A| > 1$? If $|A| < 1$, then $|x|^k \to 0$, so 0 will be a fixed point. If $|A| > 1$, $|x|^k \to \infty$, so there will not be a finite fixed point.

   When will $\displaystyle\sum_{k=0}^{N} x_k$ have a limit as $N \to \infty$? In that case, what is the (infinite) sum?

   We can rewrite

   $$\sum_{k=0}^{N} x_k = \sum_{k=0}^{N} A^k x_0 = x_0 \sum_{k=0}^{N} A^k = \frac{1 - A^{N+1}}{1 - A}$$

   so if $|A| < 1$, the infinite sum is $\frac{1}{1-A}$. If $|A| \geq 1$, the infinite sum diverges.

4. Is it possible to use a "simple" model to produce complicated behavior? Discuss in terms of what you did in our first set of exercises.

   Yes, it is possible. We saw that the same simple model,

$$x_{n+1} = kx_n(1 - x_n)$$

   can create a trajectory that goes to zero, that goes into a periodic behavior, or the trajectory might become "random", depending on the values of k and x_0.

5. Suppose that some quantity of interest involves the following variables:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

   Find a complete set of dimensionless products.

   A generic dimensionless product:

$$[\pi] = \frac{L^a}{T^a} \cdot L^b \cdot \frac{L^c}{T^{2c}} \cdot \frac{M^d}{L^{3d}} \cdot \frac{M^e}{L^e T^e}$$

   so that:

$$[\pi] = M^{d+e} L^{a+b+c-3d-e} T^{-a-2c-e}$$

   We have three equations in five unknowns, therefore two are free. Note that, because of the first equation, we cannot freely choose both d and e for the free variables. But we can choose, for example, c and e and write everything else in terms of these:

$$\begin{aligned} d &= -e \\ a + b - 3d &= -c + e \\ a &= -2c - e \end{aligned}$$

   For a complete set, choose $e = 0, c = 1$ and $e = 1, c = 0$. Substitution gives $\{a, b, c, d, e\}$ as (respectively) $\{-2, 1, 1, 0, 0\}$ and $\{-1, -1, 0, -1, 1\}$. This gives the dimensionless variables as:

$$\pi_1 = \frac{rg}{v^2}, \quad \pi_2 = \frac{\mu}{vr\rho}$$

   NOTE: You could double check this by making sure that π_1 and π_2 are dimensionless.

6. (Use a calculator) We hypothesize that $y \propto z^{1/2}$. Does the following data support this? If so, find the constant of proportionality:

y	6.4	9.1	11.1	12.8	14.3
z	3	6	9	12	15

Using a calculator, we compare y to $\sqrt{z}$ (this is one method to linearize the proportion):

y	6.4	9.1	11.1	12.8	14.3
$\sqrt{z}$	$\sqrt{3}$	$\sqrt{6}$	3	$2\sqrt{3}$	$\sqrt{15}$

If $y = k\sqrt{z}$, then $k = \frac{y}{\sqrt{z}}$, so by checking these numbers, we see that the data does support this hypothesis, and the $k \approx 3.7$.

7. If A is proportional to the square of B, how would we linearize the relationship? Same question, but $A = ke^B$?

   For $A = kB^2$, we could let $x = B^2$ so that $y = kx$. Alternatively, we could take logarithms so that $\ln(A) = \ln(k) + 2\ln(B)$. This is what we do in allometric problems.

   For the second part, we can do the same thing. Either let $x = e^B$ or take logs so that $\ln(A) - \ln(k) = B$.

8. Prove the two properties for proportions: (i) If $x \propto y$, then $y \propto x$ (ii) If $x \propto y$, $y \propto z$, then $x \propto z$

   For the first, $y \propto x$ means that $y = kx$, $k \neq 0$. Therefore, $x = \frac{1}{k}y$, and $x \propto y$.

   In the second,

   $$x = k_1 y, y = k_2 z \Rightarrow x = k_1 k_2 z \Rightarrow x = kz$$

9. Suppose that we assume that a Migration and Extinction curves are linear. Here is some data that I've "retrieved from the field"- "0" in the number of species refers to the current number, so negative values are valid. Fill in the missing numbers.

Number of Species	Rate
3	2
-1	2
-1	14/3
5/3	14/3
1.67	2.89
-0.11	2.89

   Find the equilibrium state.

   From the data, we see that the extinction rate curve is $y = -x + 3$, and the migration rate curve is $y = -\frac{2}{3}x + 4$. The point of intersection is the equilibrium, $(0.6, 3.6)$.

   Predict what happens, if in a particularly good summer, there is much more food on the island than usual. In particular, is it possible that the equilibrium does not change?

In this case, we might assume that the extinction curve decreases and the migration curve increases. To get the same equilibrium, if the extinction curve decreases, the migration curve would have to decrease as well (You might draw a picture).

10. Assume that heat loss is proportional to surface area. How much more heat loss will a 12 inch cube have than a 6 inch cube?

    The surface area of the small cube is $6 \cdot 36$. Scaling the sides by k means the new surface area is $6 \cdot (6k \cdot 6k) = k^2(6 \cdot 36)$. Therefore, double the lengths increases the surface area 4 times (and so there will also be 4 times as much heat loss).

    Now consider two irregularly shaped objects, such as submarines. If we can measure heat loss from a 7 foot scale model, how much more heat is lost from a 70 foot sub?

    The same thing happens here- An increase by a factor of 10 will mean that heat loss (which is proportional to surface area) increases by a factor of $10^2 = 100$.

11. Describe Newton's Method graphically and give the formula.

    We won't do the graph here- look to your HW for that solution. The formula is:

    $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

12. For spheres and geometrically similar boxes, show that surface area, S, is proportional to volume V to the 2/3 power.

    For spheres, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. From the first, $r = \sqrt[3]{\frac{3}{4\pi}} V^{1/3} = kV^{1/3}$. Inserting this into the second gives:

    $$S = 4\pi k^2 V^{2/3} = k_1 V^{2/3}$$

    For boxes, use a "template" box with side length 1, width w and height h (these values are fixed, and therefore constant, for the class of boxes).

    Now, given any other box in the class, the length is k, width is kw and height is kh. Therefore,

    $$S = k^2(wh + w + h) = c_1 k^2 \quad V = k^3 wh = c_2 k^3$$

    Therefore $k = \frac{1}{c_2^{1/3}} V^{1/3}$, and

    $$S = \frac{c_1}{c_2^{2/3}} V^{2/3} = c_3 V^{2/3}$$

13. Suppose that gymnasts are geometrically similar with a characteristic scaling k. First, write each of the following assumptions as mathematical statements:

(a) Ability is proportional to strength, and inversely proportional to weight.

$$A = k_1 \frac{S}{W} \text{ or } A \propto \frac{S}{W}$$

(b) Strength is proportional to muscle area.

$$S = k_2 M \text{ or } S \propto M$$

Next, how should muscle area scale with k? (Hint: Think of muscle area as proportional to surface area) How should weight scale with k? Put the 4 statements together to show that $A \propto \frac{1}{k}$.

$$M \propto k^2, \quad W \propto k^3$$

We need to relate both S and W to k. We have W, so for S,

$$S \propto M \propto k^2$$

Now,

$$A \propto \frac{S}{W} \propto \frac{k^2}{k^3} = \frac{1}{k}$$

Therefore, gymnasts should be short (this line of reasoning showed that ability is inversely proportional to height).

14. What is a "fixed point" for a difference equation of the form $x_{k+1} = f(x_k)$?

    The fixed point is the value of x so that $x = f(x)$.

    Find the fixed point(s) if $x_{k+1} = kx_k(1 - x_k)$, k is a real number.

    In this case,

    $$x = kx(1-x) \Rightarrow kx^2 - kx + x = 0 \Rightarrow x(kx + (1-k)) = 0$$

    so that $x = 0$ or $x = \dfrac{k-1}{k}$

15. State the Buckingham π Theorem. An equation is dimensionally homogeneous (or dimensionally consistent) if and only if it can be put into the form:

    $$f(\pi_1, \pi_2, \ldots, \pi_n) = 0$$

    where $\pi_1, \pi_2, \ldots, \pi_n$ form a complete set of dimensionless products.

16. What is "allometric" scaling? Allometric scaling is the term coined for biological scaling of the form $y = ax^b$.

17. What is the justification for saying that: "If $f(x, y) = 0$, then we can solve for y as a function of x: $y = g(x)$". Apply this to the equation for a unit circle. Using the unit circle, can I always solve for y in any small region about any fixed (x^*, y^*) in the domain?

The justification is the Implicit Function Theorem (you do not have to know what it is, just the name for now). For example, from the equation for a circle (in the form given from the theorem):

$$x^2 + y^2 - 1 = 0 \Rightarrow y = \pm\sqrt{1 - x^2}$$

where we would have to choose a $+$ or $-$. We cannot do this locally about any point, since the two points $(-1, 0)$ and $(1, 0)$ have vertical tangent lines, and a local region about these points would necessarily have to include points from both the upper and lower half circle.

18. Sociologists will use "Group norms" in order to talk about group behavior (how might you define a "group norm" mathematically?).

    You might just take an average of the group's intentions, however those are classified numerically.

    (i) What does our Case Study say about how a group may behave with respect to a group norm versus individual thresholds? For example, will a single individual's decision be a crucial element to a group norm? In our individual threshold model?

    We saw that a single individual threshold, if changed, can result in much different group behavior (consider our very first threshold model). In averaging over all individuals, a change in one individual will not change the entire norm by much.

    (ii) How might a group norm be effected by a set of individual thresholds, and vice versa?

    As the group dynamics progress, if people see other people doing something, they might be more willing to do that thing- so the group norm would increase (depending on how it is measured). Furthermore, if someone starts doing something (perhaps under a low threshold), the group norm might lead people to convince that person to get a higher threshold. So, in general, there might be complex interactions between group norms and individual thresholds.

19. Come up with a situation (graphically) where a set of individual thresholds produce a model that has three equilibria, one at the origin, one at $(100, 100)$ and the middle equilibrium is repelling. Your curve should be below the line $y = x$ until it crosses at the equilibrium, then be above the curve until it gets to $(100, 100)$.

    Supplementary (Extra credit) question: Such a curve (if differentiable) will necessarily have a derivative at the equilibrium that is greater than 1. Why? (This is not true in general, but for this situation it is- our graph of $f(x)$ must be monotonically increasing).

20. Same question as before, but the equilibrium is attracting.

    Reverse your curve about $y = x$.

21. The crucial element of our modeling of biological systems came from our relationship of surface area to mass, $S \propto M^{2/3}$. How did we arrive at this?

    See Question 12. In this case, if two things are geometrically similar, and S and V are the surface area and volume of the template for the class, then for a particular member of the class, its surface area $\hat{S}$ and volume $\hat{V}$ are related by:

    $$\hat{S} \propto k^2 S, \quad \hat{V} \propto k^3 V$$

    where k is the scaling factor (this assumes that the organism is three dimensional).

22. Is it possible that within any single group, we can have an affine[1] relationship with a slope of $-1/3$, but when we put the groups together, we get a relationship with slope $-1/4$? If this is true, should our model have used $-1/3$ or $-1/4$? Discuss.

    This is true (we had some data with exactly those qualities). The second part of the question gets at the heart of what we are trying to model- Our model was looking at scaling over all kinds of *different* species. But, this question remains as a big unanswered question in biology.

23. To understand how branchings scale, let us consider the following model. Take an interval, $[0, 1]$. We'll say that the first "branching" cuts the interval into two pieces, each of equal length. The second branching cuts each of those into two pieces of equal length, and so on. The following table may help. Fill in the last entry:

Branching	Number of Intervals	Width of each
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
$\vdots$	$\vdots$	$\vdots$
k	2^k	$\frac{1}{2^k}$

    Suppose we require that the last branch cover a set width, say $\frac{1}{2^{10}}$. How many branchings are required?

    $$\frac{1}{2^k} = \frac{1}{2^{10}} \Rightarrow k = 10$$

    Do the same problem, except take the initial interval to have length S.

    In this case, at stage k we cover subintervals of length $\dfrac{S}{2^k}$. In this case,

    $$\frac{S}{2^k} = \frac{1}{2^{10}} \Rightarrow k = \log_2(S) + 10$$

[1] Affine refers to $y = mx + b$ rather than $y = mx$, which is linear.

Using your previous answer, if length S is multiplied by $2^7 = 128$, how much larger is k?

$$k = \log_2(2^7 S) + 10 = 7 + (\log_2(S) + 10)$$

If length S is multiplied by c, how much larger is k?

We would add $\log_2(c)$. Note that this grows very slowly compared to c.

NOTE: It might be interesting at some point to see how this would work using volumes instead of intervals, with d branchings instead of 2.