Solutions: Review Questions

1. The solution is:

$$\boldsymbol{x} = \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

2. If AB is invertible, then by the IMT, there is a W such that $(AB)W = I$. By the rules of matrix multiplication, $A(BW) = I$. Now, by the IMT, since A is square and there is a C such that $AC = I$, A is invertible.

3. If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly dependent, then there are infinitely many (nontrivial) solutions to the equation:

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = 0$$

Apply the linear transformation T to both sides, and simplify. This shows that any solution to the equation above is also a solution to:

$$c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + c_3T(\boldsymbol{v}_3) = 0$$

which says that $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ form a linearly dependent set.

4. If $H\boldsymbol{x} = \boldsymbol{v}$ is consistent for all $\boldsymbol{v}$, then every row of H has a pivot. Therefore, there are 7 pivots. Since H has only 7 columns, there must be a pivot in every column as well. Therefore, there are no free variables, and therefore, the solution must be unique.

5. We can show the mapping is nonlinear in a couple of ways. For example, $T(-1 \cdot (1,1)) \neq -1 \cdot T(1,1)$

6. Suppose that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent. Let V be a matrix whose columns are the $\boldsymbol{v}_i$'s. Then: $V\boldsymbol{c} = \boldsymbol{0}$ has only the trivial solution. Now the question is whether or not $AV\boldsymbol{c} = \boldsymbol{0}$ has only the trivial solution. The answer is yes. Because A is invertible, we can rewrite the previous equation as $A^{-1}AV\boldsymbol{c} = A^{-1}\boldsymbol{0}$. Now this has only the trivial solution, therefore so does $AV\boldsymbol{c} = \boldsymbol{0}$.

7. False. The description given is for a *function*; that is, each x has only one y associated with it. For a function to be 1-1, the *preimage* of y must be a unique x.

8. This is a template example showing that AB may be invertible, but neither A nor B are themselves invertible. This is why it was so important in 2.3 that all matrices are square.

9. For a unique solution, $h \neq 6$, and k is anything. For infinitely many solutions, $h = 6$ and $k = 2$. For no solution, $h = 6$ and $k \neq 2$.

10. Let us say that the coefficient matrix is $m \times n$. If the system is underdetermined, then $m < n$. Therefore, at best, we can have m pivots, which leaves at least $n - m$ free variables. Therefore, if the system is consistent, it must have an infinite number of solutions.

11. Note that in each case, E needs to be 3×3 for EA to be defined.

$$E = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

12. (Graphical, see solutions in back of book)

13. Asking if the set spans $\mathbb{R}^n$ means that we are checking if every vector in $\mathbb{R}^n$ can be written as a linear combination of those vectors. If we put the vectors in the matrix A, we are asking if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$.

14. This question is basically the same as before, except we are asking for a solution *for one value* of $\boldsymbol{b}$.

15. The center of mass is $\frac{1}{10}(13, 9, 0)^T$. This is a linear combination of the "points", so it is in the span.

16. (a) $\boldsymbol{y}_1, \boldsymbol{y}_2$ must be in $\mathbb{R}^3$, and $\boldsymbol{x}_1, \boldsymbol{x}_2$ must be in $\mathbb{R}^4$. (b) $A\boldsymbol{x} = \boldsymbol{w}$ has a solution. By the matrix multiplication properties, if $\boldsymbol{w} = \boldsymbol{y}_1 + \boldsymbol{y}_2$, then $\boldsymbol{x}$ must be $\boldsymbol{x}_1 + \boldsymbol{x}_2$.

17. The two solutions are (in order):

$$\boldsymbol{x} = r\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The latter solution is a translated version of the former solution.

18. This is true. It says that the two vectors are constant multiples of each other.

19. Lots of examples. For instance, let $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \boldsymbol{v}_4 = (1,1,1,1)^T$. Let $\boldsymbol{v}_3 = (0,1,1,1)^T$.

20. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we need to satisfy the following sets of equations:

$$\begin{aligned} a + 2b &= -3 \\ c + 2d &= 0 \\ -2a + b &= 1 \\ -2c + d &= 2 \end{aligned}$$

We have a variety of ways to solve this. $a = -1, b = -1, c = -\frac{4}{5}, d = \frac{2}{5}$

21. If $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution, there must be a pivot in every column. In this case, there are n. But since A is $n \times n$, this means that there must be a pivot in every row as well. Therefore, the columns do span $\mathbb{R}^n$. In general, this is not true- That is, in general, having a pivot in every column does not imply that there is a pivot in every row.

22. See Problem 2, and let $B = A$.

23.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 13 & 7 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

24. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$, there must be a pivot in every row. Since A is square, this implies that there must be a pivot in every column. Therefore, the columns of A also span $\mathbb{R}^n$. This is not true in general.

25. False. Because $[A|I] \to [I|A^{-1}]$, the row operations that reduce A to I also "reduce" I to A^{-1}.

26. The first and second columns of AB are: $(-13, 32)^T$ and $(20, -49)^T$. The $(3,1)$ entry of B^TA is -7. We can't find the inverse of B^TB because it is not row reducible to the identity (we get one free variable, x_3).

27. (From the quiz) If $A\boldsymbol{x} = \boldsymbol{0}$, then $CA\boldsymbol{x} = C\boldsymbol{0} = \boldsymbol{0}$. Therefore, we must have that $\boldsymbol{x} = \boldsymbol{0}$.

28. The general matrix is $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, so the matrix is this case is: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

29. $T(\boldsymbol{x}) = [3\ -4\ 0\ 8]\boldsymbol{x}$

30. (From the quiz) Show that $T_1(T_2(ax + by)) = aT_1(T_2(x)) + bT_1(T_2(y))$

31. The matrix is: $\begin{pmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{pmatrix}$