Review Questions: Sections 1.1-2.3

1. Find the general solution (in vector form) to the system:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= -1 \\ -2x_1 - 6x_2 - x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 + 3x_4 &= 2 \end{aligned}$$

2. Let A, B be $n \times n$. Show that, if AB is invertible, the so is A. (Hint: There is a matrix W so that $(AB)W = I$. Why?)

3. Let T be a linear transformation. Show that if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly dependent vectors, then $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ is linearly dependent.

4. If H is 7×7 and $H\boldsymbol{x} = \boldsymbol{v}$ is consistent for every $\boldsymbol{v}$ in $\mathbb{R}^7$, then is it possible for $H\boldsymbol{x} = \boldsymbol{v}$ to have *more* than one solution for some $\boldsymbol{v} \in \mathbb{R}^7$? Why or why not?

5. Show that the mapping T below is not linear:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 3x_2 + 1 \end{bmatrix}$$

6. Let A be invertible. Show that, if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent, then $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is linearly independent. Is this true if A is not invertible?

7. True or False, and explain: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

8. Let A be given below. Find three matrices C so that $AC = I_2$.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

9. Choose h and k below so that the system has (a) no solution, (b) a unique solution, and (c) many solutions.

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ 2x_1 - hx_2 &= k \end{aligned}$$

10. A system of equations with fewer equations than unknowns is called an *undetermined* system. Suppose that such a system happens to be consistent. Explain why there must be an infinite number of solutions.

11. Let A be an 3×4 matrix. Construct the elementary matrix E to perform:
    (a) The row operation: $3r_2 + r_1 \to r_1$.
    (b) The row operation: $-3r_3$
    (c) The inverse of the row operation: $2r_1 + r_2 \to r_2$.

12. How do we approach the answer to the question: "Does the set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ span $\mathbb{R}^n$?"

13. Compare the answer of the previous problem with how we answer the question: "Is $\boldsymbol{b}$ in the span of $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$?"

14. Let A be a 3×4 matrix, let $\boldsymbol{y}_1, \boldsymbol{y}_2$ be vectors, and let $\boldsymbol{w} = \boldsymbol{y}_1 + \boldsymbol{y}_2$. Suppose that $\boldsymbol{y}_1 = A\boldsymbol{x}_1$ and $\boldsymbol{y}_2 = A\boldsymbol{x}_2$ for some vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.
    (a) What size must $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_1, \boldsymbol{x}_2$ be?
    (b) Does $A\boldsymbol{x} = \boldsymbol{w}$ have a solution? Why or why not?

15. Describe (in parametric vector form) the solution set to: $x_1 - 4x_2 + 3x_3 = 0$. Compare this with the solution set to: $x_1 - 4x_2 + 3x_3 = 7$

16. True or False (and explain): Two vectors are linearly dependent if they lie on a line through the origin.

17. Find an example to contradict the following: If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_4$ are in $\mathbb{R}^4$ and $\boldsymbol{v}_3$ is not a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4$, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly independent.

18. Suppose that:
$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
Find a matrix A so that $T(\boldsymbol{x}) = A\boldsymbol{x}$.

19. Let A be $n \times n$. If the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, do the columns of A span $\mathbb{R}^n$? Why or why not? Is your answer different if A is $n \times m$?

20. Explain why A^2 is invertible whenever A is invertible.

21. Use a matrix inverse to solve:
$$\begin{aligned} 3x_1 - 7x_2 &= -4 \\ -6x_1 + 13x_2 &= 1 \end{aligned}$$

22. Suppose that for an $n \times n$ matrix A, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b}$ in $\mathbb{R}^n$. Explain why A must be invertible. Does your answer change if A is $n \times m$?

23. True or False, and explain: If A is invertible, then the elementary row operations that reduce A to the identity also reduce A^{-1} to the identity.

24. Let A and B be given below. (a) Find the first and second columns of AB. (b) Find the $(3,1)$ entry of B^TA. (c) Determine the inverse of BB^T, if possible.
$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

25. Let A be $m \times n$. Suppose that for some matrix C, $CA = I$. Show that the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

26. Determine the matrix for the linear transformation that rotates vectors counterclockwise in $\mathbb{R}^2$ through the angle $\frac{\pi}{4}$.

27. Determine the matrix for the linear transformation T given below: $T(x_1, x_2, x_3, x_4) = 3x_1 - 4x_2 + 8x_4$

28. If T_1 and T_2 are linear transformations, show that the composition $T_1 \circ T_2$ is also a linear transformation.

29. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T(\boldsymbol{e}_1) = (1,4)$, $T(\boldsymbol{e}_2) = (-2,9)$, and $T(\boldsymbol{e}_3) = (3,-8)$. Find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.

30. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ so that $T(\mathbf{x}) = A\mathbf{x}$, where
$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix}$$
Is T 1-1? Explain. Is T onto? Explain.

31. Suppose that A, B, C are $n \times n$ matrices with B invertible, and $I - BAB^{-1} = C$. Solve this equation for A- Show your work, and if you invert a matrix, explain why it is invertible.

32. Let A be $m \times n$ with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ with $m, n > 1$. Explain why, if columns of A are linearly independent, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Your explanation should show that you know the definition of $A\mathbf{x}$ and the definition of linear independence.