Final Exam Summary Solutions

1. Given vectors $\mathbf{a}_1, \mathbf{a}_2$, the purpose of this question is to produce two vectors $\mathbf{u}_1, \mathbf{u}_2$ so that (i) $\mathbf{u}_1, \mathbf{u}_2$ are orthonormal, and (ii) $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. You might find this helpful for the Take-Home portion of the final exam.

   First, define $\mathbf{u}_1 = \dfrac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1$. (In Matlab, $\|\mathbf{v}\|$ =`norm(v)`)

   Next, define $\mathbf{u}_2 = \mathbf{a}_2 - \text{Proj}_{u_1}(\mathbf{a}_1) = \mathbf{a}_2 - \left(\mathbf{u}_1^T\mathbf{a}_2\right)\mathbf{u}_1$. Divide the result by $\|\mathbf{u}_2\|$ so that the resulting $\mathbf{u}_2$ has length 1.

   Show that Properties (i) and (ii) hold for $\mathbf{u}_1, \mathbf{u}_2$.

   SOLUTION: Property (i): Show that the resulting vectors are orthonormal. The vectors have length (or norm) 1, since we divided each by its original size. We'll show that they are orthogonal by computing the dot product:

   $$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \left(\mathbf{a}_2 - (\mathbf{u}_1^T\mathbf{a}_2)\mathbf{u}_1\right) = \mathbf{u}_1 \cdot \mathbf{a}_2 - (\mathbf{u}_1^T\mathbf{a}_2)(\mathbf{u}_1 \cdot \mathbf{u}_1) = \mathbf{u}_1^T\mathbf{a}_2 - \mathbf{u}_1^T\mathbf{a}_2 = 0$$

2. Let $U = [\mathbf{u}_1, \ldots, \mathbf{u}_k]$ be a matrix with orthonormal columns. Let W be the subspace spanned by the columns of U:

   - If $\mathbf{x} \in W$, write the *coordinates* of $\mathbf{x}$ with respect to the columns of U.

     In general, we had (Section 4.4): $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$. In that setting, the matrix $P_{\mathcal{B}}$ was invertible, so that $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$. In this case, we cannot assume that the matrix U is invertible, BUT:

     $$\mathbf{x} = U[\mathbf{x}]_U \quad \Rightarrow \quad U^T\mathbf{x} = U^TU[\mathbf{x}]_U \quad \Rightarrow \quad [\mathbf{x}]_U = U^T\mathbf{x}$$

     This last relation holds since, if a matrix has orthonormal columns, $U^TU = I$ (See the last part of this question).

   - If $\mathbf{x}$ is not contained in W, write the orthogonal decomposition of $\mathbf{x}$ in terms of W and $W^\perp$ (Hint: Orthogonal Decomposition Theorem)

     SOLUTION: By the Orthogonal Decomposition Theorem, we can write $\mathbf{x}$ uniquely as:

     $$\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$$

     where $\hat{\mathbf{x}} \in W$ and $\mathbf{z} \in W^\perp$. Since the columns of U are orthonormal, $\hat{\mathbf{x}} = \text{Proj}_U(\mathbf{x}) = UU^T\mathbf{x}$, and

     $$\mathbf{z} = \mathbf{x} - \hat{\mathbf{x}} = \mathbf{x} - UU^T\mathbf{x}$$

   - If the columns of U are in $\mathbb{R}^n$, is there any restriction on n, k (the columns are orthonormal)?

     SOLUTION: The matrix U cannot have a column of zeros (since every column has length 1), and the columns are orthogonal, so they must be linearly independent. We cannot have more than n columns (otherwise, they would be linearly dependent). Therefore, $n \geq p$ (the matrix U must be square or tall, not wide).

   - Show (by writing U in terms of its columns) that $U^TU = I$, but $UU^T \neq I$ (if U is not square). In this case, what does the map $\mathbf{x} \to UU^T\mathbf{x}$ do?

     SOLUTION: We know that $UU^T\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ into the columnspace of U, so we'll focus on the first part of the question.

     We write: $U = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_p]$, and use these to compute U^TU:

     $$U^TU = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_p^T \end{bmatrix} [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_p] = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \ldots\mathbf{u}_1^T\mathbf{u}_p & \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \ldots\mathbf{u}_2^T\mathbf{u}_p & \\ \vdots & \vdots & & \vdots \\ \mathbf{u}_p^T\mathbf{u}_1 & \mathbf{u}_p^T\mathbf{u}_2 & \ldots\mathbf{u}_p^T\mathbf{u}_p & \end{bmatrix} = I_p$$

3. A matrix A is 5×5 with two eigenvalues. One eigenspace is three dimensional and the other is two dimensional. Is A diagonalizable? Why?

The matrix is diagonalizable (Section 5.3). In face, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are three independent eigenvectors for the first eigenvalue, and $\mathbf{w}_1, \mathbf{w}_2$ are linearly independent eigenvectors for the second eigenvalue, then:

$$A = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{w}_1 \quad \mathbf{w}_2] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0\lambda_2 & \end{bmatrix} [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{w}_1 \quad \mathbf{w}_2]^{-1} = PDP^{-1}$$

4. Find the eigenvalues and eigenvectors of A and B: $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

The eigenvalues of A are $-2, 5$ with eigenvectors $[3, -4]^T$ and $[1, 1]^T$, respectively.

For B, we get: $\lambda = 3 \pm i$. The eigenvectors are $[5, 2 - i]^T, [5, 2 + i]^T$, respectively.

5. If a, b, c are distinct numbers, then the system below is inconsistent. Show that the least squares solutions can be written as $x - 2y + 5z = (a + b + c)/3$. **SOLUTION:** Write the equation in matrix form, and use the normal equations:

$$\begin{aligned} x - 2y + 5z &= a \\ x - 2y + 5z &= b \\ x - 2y + 5z &= c \end{aligned} \Rightarrow \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 5 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow A\mathbf{x} = \mathbf{b}$$

Form $A^T A\mathbf{x} = A^T\mathbf{b}$ to get:

$$\begin{bmatrix} 3 & -6 & 15 \\ -6 & 12 & -30 \\ 15 & -30 & 75 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a + b + c \\ -2(a + b + c) \\ 5(a + b + c) \end{bmatrix}$$

Row reduction on the augmented matrix gives:

$$\left[\begin{array}{ccc|c} 3 & -6 & 15 & (a + b + c) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \Rightarrow x - 2y + 5z = \frac{a + b + c}{3}$$

6. Define $T : P_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

SOLUTION: Before working with this problem, let's see what the action of T is on an arbitrary domain polynomial. An arbitrary vector in P_2 is: $p(t) = at^2 + bt + c$, and $T(p(t))$ would be:

$$T(at^2 + bt + c) = \begin{bmatrix} a - b + c \\ c \\ a + b + c \end{bmatrix}$$

(a) Find the image under T of $p(t) = 5 + 3t$. We see that $p(-1) = 5 - 3 = 2$, $p(0) = 5$ and $p(1) = 5 + 3 = 8$. The image of $5 + 3t$ is the vector $[2, 5, 8]^T \in \mathbb{R}^3$.

(b) Show that T is a linear transformation. We'll show the first one, the second is similar:

$$T(p_1(t) + p_2(t)) = T((a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)) =$$

$$\begin{bmatrix} (a_1 + a_2) - (b_1 + b_2) + (c_1 + c_2) \\ c_1 + c_2 \\ (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) \end{bmatrix} = \begin{bmatrix} a_1 - b_1 + c_1 \\ c_1 \\ a_1 + b_1 + c_1 \end{bmatrix} + \begin{bmatrix} a_2 - b_2 + c_2 \\ c_2 \\ a_2 + b_2 + c_2 \end{bmatrix} = T(p_1(t)) + T(p_2(t))$$

(c) Find the kernel of T. Does your answer imply that T is $1-1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)

The kernel of T is the set of domain values p so that $T(p) = \mathbf{0}$. From what we've written, this means that:

$$a - b + c = 0 \qquad c = 0 \qquad a + b + c = 0$$

The only solution is $a = b = c = 0$. Therefore, the kernel of T is only the zero polynomial.

Since T is linear and has only the zero in its nullspace, $T(p) = \mathbf{y}$ has at most one solution for every $\mathbf{y}$. Therefore, T must be $1-1$. Furthermore, T will be onto (See the answer to the next question).

(d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2. (This means that the matrix will act on the *coordinates* of p).

We can write T as a matrix by writing $at^2 + bt + c$ as $[a, b, c]^T$. You can verify that the matrix representation for T is:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

From this we see that the matrix is invertible, so the mapping is both 1-1 and onto (as a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$).

7. Show that if U is an $n \times k$ matrix with orthonormal columns, and $\mathbf{x}$ is in the columnspace of U, then $[x]_U$ can be written as $U^T\mathbf{x}$. (Hint: Write U in terms of its columns).

I think we've already shown this a couple of different times (See problem 2, first sub-part). Sorry about the repetition.

8. If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ compare the process of computing $[\mathbf{x}]_B$, if the vectors in $\mathcal{B}$ are linearly independent versus if they are orthonormal (You may assume that $\mathbf{x}$ is in the span of the vectors).

See the answer to Problem 2, first sub-part.

9. Show that $I - A$ is invertible when all the eigenvalues of A are less than 1 in magnitude. (Hint: What would be true if $I - A$ were not invertible?)

Using the hint, if $I - A$ were not invertible, then $\det(I - A) = 0$, and $\det(I - A) = (-1)^n \det(A - I)$, so that $\det(A - I) = 0$. This says that $\lambda = 1$ is an eigenvalue of A, but our eigenvalues are all **less than** 1.

10. Show that, if two (nonzero) vectors are orthogonal, then they are linearly independent. (Before you work out the solution, you might take this opportunity to review linear independence).

SOLUTION: Let $\mathbf{u}_1, \mathbf{u}_2$ be our two vectors. To show that they are linearly independent, we show that the only solution to:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = 0$$

is the trivial solution, $c_1 = c_2 = 0$.

Suppose the above equation is true for some values of c_1, c_2. We show that this implies that the weights must both be zero.

Take the dot product of both sides of the equation with $\mathbf{u}_1$:

$$\mathbf{u}_1 \cdot (c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = \mathbf{u}_1 \cdot \mathbf{0} = \mathbf{0}$$

$$c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{0}$$

$$c_1 + 0 = 0 \Rightarrow c_1 = 0$$

Similarly, taking the dot product of both sides of the equation with $\mathbf{u}_2$ will lead us to conclude that $c_2 = 0$.

Therefore, the only solution is the trivial solution, and the vectors are linearly independent.

NOTE: The word *NONZERO* was an important part of the question. If this had been deleted, the statement would've been false (that is, a set with the zero vector may be orthogonal, but it certainly is not linearly independent)