Final Exam Summary

Be sure to focus your studies on 5.1-5.5, and 6.1-6.3, 6.5. However, be sure you remember the earlier material as well!
Go through your old exams to be sure that you understand that material.
You won't need a calculator for the exam (no calculators allowed). Most of the computational questions will come in your take-home portion of the exam.
Here are some questions. You can also look over the multiple choice test that is also online (your exam won't be multiple choice, but it will serve as a review).

1. Given vectors $\mathbf{a}_1, \mathbf{a}_2$, the purpose of this question is to produce two vectors $\mathbf{u}_1, \mathbf{u}_2$ so that (i) $\mathbf{u}_1, \mathbf{u}_2$ are orthonormal, and (ii) $\text{Span}\,\{\mathbf{a}_1, \mathbf{a}_2\} = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$. You might find this helpful for the Take-Home portion of the final exam.

   First, define $\mathbf{u}_1 = \dfrac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1$. (In Matlab, $\|\mathbf{v}\| =$`norm(v)`)

   Next, define $\mathbf{u}_2 = \mathbf{a}_2 - \text{Proj}_{u_1}(\mathbf{a}_1) = \mathbf{a}_2 - \left(\mathbf{u}_1^T\mathbf{a}_2\right)\mathbf{u}_1$. Divide the result by $\|\mathbf{u}_2\|$ so that the resulting $\mathbf{u}_2$ has length 1.

   Show that Properties (i) and (ii) hold for $\mathbf{u}_1, \mathbf{u}_2$.

2. Let $U = [\mathbf{u}_1, \ldots, \mathbf{u}_k]$ be a matrix with orthonormal columns. Let W be the subspace spanned by the columns of U:

   - If $\mathbf{x} \in W$, write the *coordinates* of $\mathbf{x}$ with respect to the columns of U.
   - If $\mathbf{x}$ is not contained in W, write the orthogonal decomposition of $\mathbf{x}$ in terms of W and $W^\perp$ (Hint: Orthogonal Decomposition Theorem)
   - If the columns of U are in $\mathbb{R}^n$, is there any restriction on n, k (the columns are orthonormal)?
   - Show (by writing U in terms of its columns) that $U^TU = I$, but $UU^T \neq I$ (if U is not square). In this case, what does the map $\mathbf{x} \rightarrow UU^T\mathbf{x}$ do?

3. A matrix A is 5×5 with two eigenvalues. One eigenspace is three dimensional and the other is two dimensional. Is A diagonalizable? Why?

4. Find the eigenvalues and eigenvectors of A and B: $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

5. If a, b, c are distinct numbers, then the system below is inconsistent. Show that the least squares solutions can be written as $x - 2y + 5z = (a + b + c)/3$.

$$\begin{aligned} x - 2y + 5z &= a \\ x - 2y + 5z &= b \\ x - 2y + 5z &= c \end{aligned}$$

6. Define $T : P_2 \rightarrow \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

   (a) Find the image under T of $p(t) = 5 + 3t$.

   (b) Show that T is a linear transformation.

   (c) Find the kernel of T. Does your answer imply that T is $1 - 1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)

(d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2. (This means that the matrix will act on the *coordinates* of p).

7. Show that if U is an $n \times k$ matrix with orthonormal columns, and $\mathbf{x}$ is in the columnspace of U, then $[x]_U$ can be written as $U^T\mathbf{x}$. (Hint: Write U in terms of its columns).

8. If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ compare the process of computing $[\mathbf{x}]_B$, if the vectors in $\mathcal{B}$ are linearly independent versus if they are orthonormal (You may assume that $\mathbf{x}$ is in the span of the vectors).

9. Show that $I - A$ is invertible when all the eigenvalues of A are less than 1 in magnitude. (Hint: What would be true if $I - A$ were not invertible?)

10. Show that, if two (nonzero) vectors are orthogonal, then they are linearly independent. (Before you work out the solution, you might take this opportunity to review linear independence).