Summary: Making Nonlinear Functions into (Locally) Linear Functions

We denote a nonlinear function $F : \mathbb{R}^n \to \mathbb{R}^m$ by its coordinate functions,

$$\boldsymbol{y} = F(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^1$. The Jacobian of F is an $m \times n$ array of partial derivatives:

$$J(F(\boldsymbol{x})) = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & & & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

Note that the i^{th} row of $J(F)$ is the gradient of f_i.
From Calculus I, if $y = f(x)$, then the local linearization of f at $(a, f(a))$ is the equation of the tangent line,

$$y - f(a) = f'(a)(x - a)$$

Now, if $F : \mathbb{R}^n \to \mathbb{R}^m$, the local linearization of F about a vector $\boldsymbol{a}$ is:

$$\boldsymbol{y} - F(\boldsymbol{a}) = J(F(\boldsymbol{a}))(\boldsymbol{x} - \boldsymbol{a})$$

If we let $\boldsymbol{v} = \boldsymbol{y} - F(\boldsymbol{a})$ and $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{a}$, then this is:

$$\boldsymbol{v} = J(F(\boldsymbol{a}))\boldsymbol{v} \text{ or "}A\boldsymbol{x} = \boldsymbol{b}''$$

Special Cases:

- If $F : \mathbb{R}^1 \to \mathbb{R}^n$ (these are "parametric equations", or "space curves"), then the Jacobian collapses to a single column of regular derivatives, and the tangent line at a is:

$$\boldsymbol{y} - F(a) = \begin{bmatrix} f_1'(a) \\ f_2'(a) \\ \vdots \\ f_n'(a) \end{bmatrix} (t - a)$$

- If $F : \mathbb{R}^n \to \mathbb{R}^1$, then the Jacobian collapses to a single row (to the gradient of F), and the local linearization is the tangent plane:

$$y - F(\boldsymbol{a}) = [\, f_{x_1}(a) \;\; f_{x_2}(a) \;\; \ldots \;\; f_{x_n}(a) \,](\boldsymbol{x} - \boldsymbol{a})$$

  (Note the sizes of things here!)

EXERCISES:

1. Find the general Jacobian matrix for each function

   - $F: \mathbb{R}^2 \to \mathbb{R}^3$ where:
   $$F(x, y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ y^2 - 3xy \end{bmatrix}$$
   - $G: \mathbb{R}^3 \to \mathbb{R}^2$ where:
   $$F(x, y, z) = \begin{bmatrix} x^2 - y^2 \\ 3xyz - 5 \end{bmatrix}$$

2. For the first function, F, in the previous problem, find the linearization of F at $\boldsymbol{x} = (0, 1)$.

   Use this to approximate $F(1/2, 1/2)$, and check your answer by actually computing $F(1/2, 1/2)$.

3. For the second function, G, in Problem 1, find the linearization of G at $\boldsymbol{x} = (1, 1, 1)$.

   Use this to approximate $G((1, 0, 1))$ and check your answer by computing $G((1, 0, 1))$.

4. Is the local linear map for F 1-1? Onto?

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6. Let $F: \mathbb{R}^1 \to \mathbb{R}^3$ by:
   $$F(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$
   Find an equation for the tangent line to the graph of F at $t = \pi/2$.

7. Let $F: \mathbb{R}^3 \to \mathbb{R}^1$ by:
   $$F(x, y, z) = 3xy - y^2 + 4 - yz$$
   Find an equation for the tangent "plane" to the graph of F at $(1, 1, 1)$ (the quotes are included because the graph of F is in 4 dimensions).

8. For the given function $F: \mathbb{R}^2 \to \mathbb{R}^2$, first solve for (x, y) such that $F(x, y) = \mathbf{0}$, then linearize F about these points:
   $$F(x, y) = \begin{bmatrix} x(3 - x - 2y) \\ y(2 - x - y) \end{bmatrix}$$
   Note: You need to solve these systems simultaneously. For example, for the first coordinate to be zero, we need either $x = 0$ or $x = 3 - 2y$. Use the second equation now to solve for the corresponding y's.