Solutions to "Nonlinear Functions" handout, 1-7

EXERCISES:

1. Find the general Jacobian matrix for each function

   - $F : \mathbb{R}^2 \to \mathbb{R}^3$ where:

     $$F(x, y) = \begin{bmatrix} x^2 + \sin(x) \\ x(y-2) \\ y^2 - 3xy \end{bmatrix}$$

     The Jacobian matrix is:

     $$JF(x, y) = \begin{bmatrix} 2x + \cos(x) & 0 \\ y - 2 & x \\ -3y & 2y - 3x \end{bmatrix}$$

   - $G : \mathbb{R}^3 \to \mathbb{R}^2$ where:

     $$G(x, y, z) = \begin{bmatrix} x^2 - y^2 \\ 3xyz - 5 \end{bmatrix}$$

     The Jacobian is:

     $$JG(x, y, z) = \begin{bmatrix} 2x & -2y & 0 \\ 3yz & 3xz & 3xy \end{bmatrix}$$

2. For the first function, F, in the previous problem, find the linearization of F at $\boldsymbol{x} = (0, 1)$.

   First, $F(0, 1)$ is found by substitution to be: $(0, 0, 1)^T$. Next, evaluate the Jacobian matrix for $x = 0$, $y = 1$:

   $$JF(0, 1) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -3 & 2 \end{bmatrix}$$

   The linearization is:

   $$\boldsymbol{y} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -3 & 2 \end{bmatrix} \left(\boldsymbol{x} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

   Use this to approximate $F(1/2, 1/2)$, and check your answer by actually computing $F(1/2, 1/2)$.

   The approximation to $F(1/2, 1/2)$ is:

   $$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -3/2 \end{bmatrix}$$

   The direct computation gives approximately: $(0.73, -0.75, -1/2)$

3. For the second function, G, in Problem 1, find the linearization of G at $\boldsymbol{x} = (1, 1, 1)$.

   First, $G(1, 1, 1)$ is found by substitution to be: $(0, -2)^T$. Next, evaluate the Jacobian matrix for $x = 1$, $y = 1$, $z = 1$:

$$JG(1, 0, 1) = \begin{bmatrix} 2 & -2 & 0 \\ 3 & 3 & 3 \end{bmatrix}$$

   The linearization is:

$$\boldsymbol{y} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \left(\boldsymbol{x} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

   Use this to approximate $G((1, 0, 1))$ and check your answer by computing $G((1, 0, 1))$.

   First, $G(1, 0, 1) = (1, -5)^T$. The approximation is:

$$\boldsymbol{y} = \begin{bmatrix} 2 & -2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

4. Is the local linear map for F 1-1? Onto?

   Since the RREF of the matrix shown has a pivot in every column, the map is one to one. Since there is not a pivot in every row, it is not onto.

5. Is the local linear map for G 1-1? Onto?

   Since the RREF of the matrix has a pivot in every row, the map is onto. Since there is not a pivot in every column, the map is not one to one.

6. Let $F : \mathbb{R}^1 \to \mathbb{R}^3$ by:

$$F(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

   Find an equation for the tangent line to the graph of F at $t = \pi/2$.

   We'll write the equation for the tangent line by using the linearization,

$$\boldsymbol{y} - F(\pi/2) = F'(\pi/2)(t - \pi/2)$$

   We see that $F(\pi/2) = (0, 1, \pi/2)^T$, and $F'(\pi/2) = (cos(\pi/2), -\sin(\pi/2), 1)^T = (0, -1, 1)^T$. Therefore, the line has the equation:

$$\boldsymbol{y} - \begin{bmatrix} 0 \\ 1 \\ \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} (t - \pi/2)$$

which could be re-written as:

$$\boldsymbol{y} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 + \frac{\pi}{2} \\ 0 \end{bmatrix}$$

7. Let $F : \mathbb{R}^3 \to \mathbb{R}^1$ by:

$$F(x, y, z) = 3xy - y^2 + 4 - yz$$

Find an equation for the tangent "plane" to the graph of F at $(1, 1, 1)$ (the quotes are included because the graph of F is in 4 dimensions).

We note that $F(1, 1, 1) = 5$.

We use the linearization where now the Jacobian is simply the gradient,

$$JF = \nabla F = [3y \quad 3x - 2y - z \quad -y]$$

At $(1, 1, 1)$, $\nabla F = [1, 0, -1]$. We're using (x, y, z) as elements of F, so we'll let w be the new dependent variable:

$$w - 5 = [1, 0, -1] \begin{bmatrix} x - 1 \\ y - 1 \\ z - 1 \end{bmatrix}$$

Which could also be written as:

$$w - 5 = (x - 1) - (z - 1)$$

or:

$$0 = (x - 1) - (z - 1) - (w - 5)$$

8. For the given function $F : \mathbb{R}^2 \to \mathbb{R}^2$, first solve for (x, y) such that $F(x, y) = \mathbf{0}$, then linearize F about these points:

$$F(x, y) = \begin{bmatrix} x(3 - x - 2y) \\ y(2 - x - y) \end{bmatrix}$$

Note: You need to solve these systems simultaneously. For example, for the first coordinate to be zero, we need either $x = 0$ or $x = 3 - 2y$. Use the second equation now to solve for the corresponding y's.