EXAMPLE: Section 1.1, Problems 12, 13 and 28

12. Solve the system:

$$\begin{array}{rrrcr} x_1 & -3x_2 & +4x_3 & = & -4 \\ 3x_1 & -7x_2 & +7x_3 & = & -8 \\ -4x_1 & +6x_2 & -x_3 & = & 7 \end{array}$$

SOLUTION:

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array}\right] \overset{\substack{-3r_1+r_2\to r_2 \\ 4r_1+r_3\to r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right]$$

At this point, we could divide row 2 by 2, but that would introduce fractions. Alternatively, keep that for now.

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right] \overset{3r_2+r_3\to r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Because the last row is translated to read: "$0 = 3$", this system is inconsistent (has no solution).

13. Solve the system:

$$\begin{array}{rrrcr} x_1 & & -3x_3 & = & 8 \\ 2x_1 & +2x_2 & +9x_3 & = & 7 \\ & x_2 & +5x_3 & = & -2 \end{array}$$

SOLUTION:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \overset{-2r_1+r_2\to r_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right] \overset{r_3\leftrightarrow r_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right] \overset{-2r_2+r_3\to r_2}{\longrightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right] \overset{\frac{1}{5}r_2\to r_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{\substack{-5r_3+r_2\to r_2 \\ 3r_3+r_1\to r_1}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, the solution is unique, and

$$x_1 = 5, x_2 = 3, x_3 = -1$$

(EXTRA PRACTICE: Verify that this is the solution by substitution of these numbers into the original equations!)

28. Find a, b, c, d so that the system below is consistent for all values of f and g. Assume that $a \neq 0$.

$$\begin{aligned} ax + by &= f \\ cx + dy &= g \end{aligned}$$

SOLUTION: Form the augmented matrix:

$$\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right]$$

Multiply the first row by $-\dfrac{c}{a}$ (which is always possible since $a \neq 0$), and add to the second row:

$$\left[\begin{array}{cc|c} a & b & f \\ 0 & d - \frac{bc}{a} & g - \frac{fc}{a} \end{array}\right]$$

For this system to be consistent for *any* f, g, we must have that

$$d - \frac{bc}{a} \neq 0 \;\Rightarrow\; \frac{ad - bc}{a} \neq 0 \;\Rightarrow\; ad - bc \neq 0$$

Side Remark 1: If $d - \frac{bc}{a} = 0$, it is *possible* to have a consistent system- if $g - \frac{fc}{a}$ is also 0. But in this case, the system is not consistent for *all* f, g.

Side Remark 2: You might recall that $ad - bc$ is a special number for the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

. It is called the *determinant*.