A New Class of Vector Spaces: $\mathbb{Z}_k^n$

1. We define $\mathbb{Z}_k$ as the set of nonnegative integers from 0 to $k-1$. For example, $\mathbb{Z}_2 = \{0, 1\}$.

2. We can make $\mathbb{Z}_k$ into a vector space by defining addition and scalar multiplication (with scalars from $\mathbb{Z}_k$) by:

   - Addition: If $a+b=c$, the result is the remainder after division by k. For example, in $\mathbb{Z}_2$, $1+1=2$ which has a zero remainder after division by 2. Therefore, $1+1=0$. Another example: What is $3+5$ in $\mathbb{Z}_7$? Since $3+5=8$, and division by 7 gives a remainder of 1, then $3+5=1$ in $\mathbb{Z}_7$. Another way to say this: $3+5=1 \bmod 7$ Try these examples[1]: What is $4+1 \bmod 6$? What is $4+5 \bmod 7$?
   - Scalar multiplication proceeds in the same way[2]. Try these: What is $3 \cdot 4 \bmod 5$? What is $2 \cdot 3 \bmod 4$? What is $2 \cdot 2 \bmod 3$?

3. We define $\mathbb{Z}_k^n$ as an $n-$tuple of numbers (usually a column of n numbers), where the values are taken from $\mathbb{Z}_k$.

4. We looked at matrices with columns coming from $\mathbb{Z}_k^m$. For example, we could consider the following matrix with columns in $\mathbb{Z}_2^3$ (the rows are in $\mathbb{Z}_2^4$)

   $$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

   Find a basis for the column space of A:

   Row reduction would proceed as usual, with "mod 2" arithmetic and scalar multiplication:

   $$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \overset{r_1+r_3 \to r_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \overset{r_2+r_3 \to r_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{r_3+r_2 \to r_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   So the basis for the column space:

   $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

   In fact, we could list all elements of the column space. This would be all possible linear combinations of these 3 columns using weights of either $0, 1$. (There would be 8 elements in the column space).

   Using the same matrix, let's find a basis for the nullspace. We already have the RREF of A, so write out the equations:

   $$\begin{aligned} x_1 &= -x_4 \\ x_2 &= -x_4 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned} \Rightarrow x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \overset{\text{mod } 2}{=} x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   In this example, there are only two vectors in the nullspace.

[1] Answers: 5, 2

[2] Answers: 2, 2, 1

5. Try this using mod 7 arithmetic: Find the RREF of A mod 7. The answer is given, but be sure you can do it, too! To assist us, the multiplication and addition tables are below:

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

·	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

$$A = \begin{bmatrix} 3 & 2 & 5 & 1 \\ 3 & 1 & 6 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

To get you started, first create a leading "1" by multiplying row 1 by 5.

$$\begin{bmatrix} 3 & 2 & 5 & 1 \\ 3 & 1 & 6 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \overset{5r_1 \to r_1}{\longrightarrow} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 1 & 6 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \overset{4r_1+r_2 \to r_2}{\longrightarrow} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 6 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \overset{r_2+r_3 \to r_2}{\longrightarrow} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \overset{5r_3 \to r_3}{\longrightarrow} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \overset{\substack{6r_3+r_2 \to r_2 \\ 3r_3+r_1 \to r_1}}{\longrightarrow} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 6 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix} \overset{6r_2 \to r_2}{\longrightarrow} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \overset{4r_2+r_1 \to r_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

The columnspace is now straightforward. To find the nullspace, note that in mod 7 arithmetic, "-1" is 6, "-5" is 2, and "-6" is 1.

Try another matrix mod 7 on your own! You can check your answer using Maple- For the previous example, here are the Maple commands:

```
with(linalg):
A:=matrix(3,3,[[3,2,5,1],[3,1,6,2],[0,1,2,3]]);
Gausselim(A)mod 7
Gaussjord(A)mod 7
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