Final Exam Survey of Questions

Note: Your final exam will not be multiple choice. This overview is meant to give you a wide range of questions. You should be sure that you thoroughly understand the underlying concepts behind each question/answer.

1. If A is a 5×8 matrix and B is a 8×6 matrix, and Z is a 6×5 zero matrix, which of the following statements is true?
   (a) $A + Z = A$
   (b) $AB = BA$
   (c) $BZ = Z$
   (d) ZA is a 6×8 zero matrix.

2. If A and B are $n \times n$ matrices, and $AB = -BA$, then $A^2 + B^2 =$?
   (a) $(A + B)(A - B)$
   (b) $(A - B)^2$
   (c) $(A - B)(A + B)$
   (d) $(A + B)(-A - B)$
   (e) None of these.

3. The row-reduced form of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ is
   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & 4 & 3 & 1 \\ 2 & 4 & 0 & 5 \\ 7 & 8 & 9 & 8 \end{bmatrix}$, then the system of linear equations $A\boldsymbol{x} = 0$, where $\boldsymbol{x}$ is 4×1 has how many solutions?
   (a) Has infinitely many solutions.
   (b) Has one solution.
   (c) Has no solution.

5. If A is a 3×5 matrix, $\boldsymbol{b}$ is a 3×1 matrix and the rank of A is 3, then $A\boldsymbol{x} = \boldsymbol{b}$ has how many solutions?
   (a) Has infinitely many solutions.
   (b) Has one solution.
   (c) Has no solution.
   (d) Can't be determined, need more information.

6. If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, and A is 4×4 and $\boldsymbol{b}$ is 4×1, which of the following statements is False?
   (a) $\det(A) \neq 0$
   (b) rank of $A =$ rank of $[A|B]$
   (c) $A\boldsymbol{x} = 0$ has a non-trivial solution
   (d) $A\boldsymbol{x} = 0$ has a unique solution
   (e) $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

7. Which of the following sets is linearly dependent?
   (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$
   (b) $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \end{bmatrix} \right\}$
   (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\}$
   (d) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$
   (e) None of these

8. If the row reduced form of a matrix A is $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, then the solution set of $A\boldsymbol{x} = 0$ is:
   (a) $\{(x_1, x_2, x_3, x_4) \mid x_1 = 3x_2, x_3 = -2x_4\}$
   (b) $\{(x_1, x_2, x_3, x_4) \mid x_1 = 3, x_3 = -2\}$
   (c) $\{(x_1, x_2) \mid x_1 - 3x_2 = 0, x_1 + 2x_2 = 0\}$
   (d) $\{(x_1, x_2) \mid x_1 = 3, x_2 = 2\}$
   (e) none of the above.

9. If A, B, C are square invertible matrices and $ABC = I$, then B^{-1} is:
   (a) AC
   (b) $A^{-1}C^{-1}$
   (c) CA
   (d) $C^{-1}A^{-1}$

(e) None of these.

10. Which of the following forms an orthonormal basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{3}} \\ \frac{-2}{\sqrt{3}} \\ \frac{-2}{\sqrt{3}} \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{bmatrix} \right\}$

(e) None of these.

11. If $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$, and $\det(A) = 10$, then the determinant of: $\begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ 2b_1 & 2b_2 & 2b_3 & 2b_4 \\ 3a_1 & 3a_2 & 3a_3 & 3a_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$ is:

(a) $(2^4)(3^4)(10)$

(b) -60

(c) 60

(d) $-(2^4)(3^4)(10)$

(e) None of these.

12. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 4 \\ 7 & 0 & 5 \end{bmatrix}$, which of the following statements is true?

(a) $\det(A) = 0$

(b) $\text{rank}(A) = 2$

(c) $\det(A^T) = 5$

(d) $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for every 3×1 vector $\boldsymbol{b}$.

(e) all of these statements are false.

13. Which of the following is the characteristic equation of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$?

(a) $\lambda^2 - 2\lambda - 3 = 0$

(b) $\lambda^2 - 2\lambda + 5 = 0$

(c) $\lambda^2 - 2\lambda + 3 = 0$

(d) $\lambda^2 - 4 = 0$

(e) None of these.

14. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Which of the following is similar to A?

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$

(e) None of these.

15. Which of the following is a subspace of P_3 (the space of polynomials of degree less than or equal to 3)?

(a) $\{a_0 + a_1x + x^2 \mid a_0, a_1 \text{ are real}\}$

(b) $\{2 + a_1x + a_2x^2a_3x^3 \mid a_1, a_2, a_3 \text{ are real}\}$

(c) $\{0, -1, 1\}$

(d) $\{1, x, x^2, x^3\}$

(e) None of the above.

16. Which of the following sets span $\mathbb{R}^3$?

(a) $\{(1,1,1), (-1,0,1)\}$

(b) $\{(1,2,1), (2,1,3), (2,4,2)\}$

(c) $\{(2,2,2), (1,0,1), (3,2,3)\}$

(d) $\{(1,0,0), (0,-1,0), (0,0,1), (1,-1,1)\}$

(e) None of the above.

17. The coordinates of $p(x) = 5 + 3x + x^2$ relative to the basis $\{1 + x, 2 - x^2, 3 + x + x^2\}$ is

(a) $(-1, 1, 2)$

(b) $(2, 0, 1)$

(c) $(0, 1, 1)$

(d) $(-1, 2, 0)$

(e) None of these.

18. Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 4 & -2 & 1 \end{bmatrix}$ Then the dimensions of the row space, column space and nullspace of A are, respectively:

(a) 3, 4, 2

(b) 2, 2, 2

(c) 1, 2, 1

(d) 3, 3, 1

(e) None of these.

19. Find all eigenvectors of $A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ -2 & 2 & 1 \end{bmatrix}$ which correspond to $\lambda = 1$.

(a) $k_1 \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix}$, $k \neq 0$

(b) $k_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $k \neq 0$

(c) $k_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $k_1, k_2 \neq 0$

(d) $k_1 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $k_1, k_2, k_3 \neq 0$

(e) None of the above.

20. Which of the following is a linear transformation from R^3 to R^2?

(a) $T(a, b, c) = (a^2, 0)$

(b) $T(x) = Ax$, where $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 4 \end{bmatrix}$

(c) $T(a, b, c) = (a + b, a - c)$

(d) $T(a, b, c) = (2, b, c)$

(e) None of the above.

21. Let $T : R^2 \to R^2$ be a linear transformation and $T(1, 0) = (1, -1)$ and $T(0, 1) = (0, 2)$. Then what is the image of $(3, 1)$, and what is the preimage of $(0, 4)$?

(a) image=$(1, 3)$ preimage=$(-2, 2)$

(b) image=$(3, -1)$ preimage=$(0, 2)$

(c) image=$(-3, -1)$ preimage=$(-2, 2)$

(d) image=$(0, 0)$ preimage=$(-1, 0)$

(e) None of the above.

22. If A, B, C are nonsingular $n \times n$ matrices, which of the following is false?

(a) $(A + BC)^T = A^T + C^T B^T$

(b) $(C^{-1}AB)^{-1} = B^{-1}A^{-1}C$

(c) $A(B + 3C) = AB + 3AC$

(d) $\det(A^{-1}B^T) = \left(\frac{1}{\det(A)}\right)\left(\frac{1}{\det(B)}\right)$

23. If $\begin{cases} 2x + 3y - z & = & 1 \\ x + 2y + 3z & = & 4 \\ 5x - 3y - 4z & = & 10 \end{cases}$

and $\Delta = \det \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 3 \\ 5 & -3 & -4 \end{bmatrix}$, then $z =$

(a) $\dfrac{\det \begin{bmatrix} 1 & 3 & -1 \\ 4 & 2 & 3 \\ 10 & -3 & -4 \end{bmatrix}}{\Delta}$

(b) $\dfrac{\det \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ 5 & 10 & -4 \end{bmatrix}}{\Delta}$

(c) $\dfrac{\det \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 8 \\ 5 & -3 & 20 \end{bmatrix}}{\Delta}$

(d) $\dfrac{\det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 5 & -3 & 10 \end{bmatrix}}{\Delta}$

(e) None of the above.

24. Let A be a 2×2 matrix, and let one of the eigenvalues be $\lambda = 3 - i$, with corresponding eigenvector $\begin{bmatrix} 2 - i \\ 1 \end{bmatrix}$. Then $A = PCP^{-1}$, where P, C are, respectively:

(a) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 - i & 0 \\ 0 & 2 + i \end{bmatrix}$

(d) None of the above.

25. Which of the following statements is TRUE? You may assume that all matrices are $n \times n$.

(a) A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

(b) If R^n has a basis of eigenvectors of A, then A is diagonalizable.

(c) A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

(d) If A is diagonalizable, then A is invertible.

26. Which of the following is FALSE? (More than one possible)

(a) If b is in Col(A), then every solution to $Ax = b$ is a least squares solution.

(b) The least squares solution to $Ax = b$ is the point in the column space of A closest to b.

(c) A least squares solution to $Ax = b$ is a list of weights that, when applied to the columns of A, produces the orthogonal projection of b onto $\text{Col}(A)$.

(d) If $\hat{x}$ is a least squares solution of $Ax = b$, then $\hat{x} = (A^T A)^{-1} A^T b$.

27. True or False, and explain:

(a) If U is a square matrix so that $U^T U = I$, then $\det(U) = 1$.

(b) If the columns of A are linearly dependent, then $\det(A) = 0$.

(c) $\det(A + B) = \det(A) + \det(B)$

(d) If $\boldsymbol{x}$ is orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$, then $\boldsymbol{x}$ must be orthogonal to $\boldsymbol{u} - \boldsymbol{v}$

(e) If V is spanned by k vectors, $\dim(V) = k$.

(f) If A contains a row or column of zeros, then 0 is an eigenvalue of A.

(g) The orthogonal projection of $\boldsymbol{y}$ onto $\boldsymbol{u}$ is a scalar multiple of $\boldsymbol{y}$.

(h) If A is 5×5, and A has fewer than 5 distinct eigenvalues, then A is not diagonalizable.

(i) If $\hat{y} = \text{Proj}_{\boldsymbol{u}}(\boldsymbol{y})$, then $y - \hat{y}$ is orthogonal to $\boldsymbol{u}$.

(j) If U is an $m \times n$ matrix with orthonormal columns, then $U^T U = I$.

(k) Similar matrices have the same eigenvalues.

(l) If U is an $m \times n$ matrix with orthonormal columns, then $UU^T = I$.

(m) Similar matrices have the same eigenvectors.

(n) If B is an echelon form of A, then the pivot columns of B form a basis for $\text{Col}(A)$.

(o) The sum of two eigenvectors is an eigenvector.

(p) If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

(q) If $BC = BD$, then $C = D$.

(r) If there is a $\boldsymbol{b}$ so that $A\boldsymbol{x} = \boldsymbol{b}$ is not consistent, then the transformation $\boldsymbol{x} \to A\boldsymbol{x}$ is not one to one.

(s) If A is invertible, then the columns of A^{-1} are linearly independent.

(t) If A is $n \times n$ and invertible, then A^T is invertible.

(u) If AB is invertible, then $(AB)^{-1} = B^{-1} A^{-1}$.

(v) The set of vectors orthogonal to a fixed vector is a subspace.

(w) If A is square and A has linearly independent columns, then it also has linearly independent rows.

(x) $AB = [A\boldsymbol{b}_1 + A\boldsymbol{b}_2 + \ldots + A\boldsymbol{b}_n]$

28. If vector space V has a finite basis of n vectors, then we know that V is isomorphic to $\mathbb{R}^n$. Write down the isomorphism $T : V \to \mathbb{R}^n$.