Solutions to the Sample Final

1. (a) is false, since $A + Z$ is not defined. (b) is false because BA is not defined. (c) is false because BZ is size 8×5, and Z is 6×5. (d) is the correct response.

2. (a) is false, and is equal to $A^2 - AB + BA + B^2 = A^2 - 2AB + B^2$, with the extra information. (b) is True, since $A^2 - BA - AB + B^2 = A^2 - BA + BA + B^2$, with the extra info. (c) is false for the same reasone that (a) is false, and (d) is false.

3. You should find that (b) is the row-reduced form of A.

4. Since A is 3×4, and there is one solution, $X = 0$, then we must have at least one free variable. We therefore have an infinite number of solutions (choice (a)).

5. If A has rank 3, there are three pivots, and so every row of A has a pivot. Therefore, at least one solution exists. Furthermore, we must have 2 free variables, so that the solution is not unique. The choice is therefore (a).

6. If $AX = B$ has a unique solution, then A is invertible. Therefore, (a) is true. Furthermore, if A is 4×4, then A has 4 pivots, one in each row. Therefore, the rank of A is the same as the rank of $[A|B]$. Choice (c) is False, by the Invertible Matrix Theorem (so by the same theorem, (d) and (e) are True).

7. (a) is not linearly dependent. (b) is linearly dependent, since there are more vectors than entries in a vector. (c) is not linearly dependent (if you form a matrix from the vectors, the matrix would be upper triangular), and (d) is not linearly dependent.

8. The variables x_2 and x_3 are free, so a little checking would show that (a) is the answer.

9. Choice (c) is the answer, since: $ABC = I \Rightarrow B = A^{-1}C^{-1} \Rightarrow B^{-1} = (A^{-1}C^{-1})^{-1} \Rightarrow B^{-1} = CA$.

10. Choice (a) is not, since the second and third vectors are not orthogonal. The vectors in (b) are not of unit length. The third vector in (c) does not have unit length. Some checking shows that (d) is orthonormal.

11. First, recall that switching rows means that the determinant is multiplied by (-1), and that multiplying a single row multiplies the determinant (contrast this with multiplying a whole matrix by a constant). The new matrix is obtained from the old by first switching rows a and c, then c and d. Since there are two row switches, we multiply by $(-1)^2 = 1$. Row 2 is multiplied by 2, which multiplies the determinant. Row 3 is multiplied by 3, which multiplies the determinant. Therefore, the determinant is $2 \cdot 3 \cdot 10 = 60$, which is choice (c).

12. In this case, we find that the $\det(A) = -18$, so A is full rank, and $\det(A^T) = -18$. Since A is invertible, part (d) is true.

13. A short computation should verify that (a) is true.

14. For this problem, we should be reminded of DIAGONALIZATION. We could restate this problem as: $A = PDP^{-1}$ for some matrix P and diagonal matrix D. What is the matrix D? So, we find the eigenvalues (which are $3, 0$). Therefore, the answer is (c).

15. This problem asks us to recall the three parts to the definition of a subspace. That is, the set contains the zero vector, and is closed under addition and scalar multiplication. Choices (a) and (b) don't satisfy any of the parts to be a subspace. Choice (c) contains only three elements. This is a little tricky, since it does contain zero, and is closed under addition. It is not closed under scalar multiplication, however. Choice (d) is our standard basis for P_3.

16. Choice (a) does not, since there are only two vectors. Choice (b) does not, since the third vector is twice the first. Choice (c) does not, since the sum of the first and second vector is the third. Choice (d), while not a basis, has three linearly independent vectors in R^3, so must span R^3.

17. We want to solve the following for a, b, c:

$$5 + 3x + x^2 = c_1(1 + x) + c_2(2 - x^2) + c_3(3 + x + x^2)$$

which results in solving the system:

$$\begin{aligned} a + 2b + c &= 5 \\ a + 0 + c &= 3 \\ 0 - b + c &= 1 \end{aligned}$$

Solve this to find choice (b) is correct.

18. Row reduce A until you get that the rank is 3. This automatically gives choice (d), since the rank is the dimension of the column space and the dimension of the row space.

19. Put $A - \lambda I$ into row reduced echelon form to find that (a) is the correct choice.

20. Choice (a) is not correct, since the a^2 term makes T nonlinear. Choice (b) is incorrect, since it represents a mapping from R^2 to R^3. Choice (c) is correct- the matrix representation is:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Choice (d) again is not linear (T must map zero to zero).

21. The matrix representation for T is

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

so the image of $(3, 1)$ is

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

and the preimage of $(0, 4)$ is

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

22. (a),(b), and (c) are all true. (d) is false, since $\det(B^T) = \det(B)$, not $\frac{1}{\det(B)}$

23. By Cramer's Rule, the correct choice is (d). Just to be sure nothing tricky is going on, you should also verify that $\Delta \neq 0$.

24. The correct choice is (b). See Theorem 9, p. 384.

25. (This is Exercise 21, p. 320). Part (b) is the only true statement.

26. Parts (b) and (d) are False. Part (b) because the solution is the $\hat{x}$, not the $\hat{b}$, which is what part (b) refers to. Part (d) is False because it is true ONLY if A has full rank (otherwise, $A^T A$ is not invertible).

27. True or False, and explain:

    (a) False. $U^T U = I \Rightarrow \det(U) \cdot \det(U^T) = (\det(U))^2 = 1$, so $\det(U) = \pm 1$.

    (b) True. If the columns are linearly dependent, then the matrix is not invertible, so the determinant is zero.

    (c) False. We cannot split the determinant this way (Can you show a counterexample?)

    (d) True: $(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{x} = \boldsymbol{u} \cdot \boldsymbol{x} - \boldsymbol{v} \cdot \boldsymbol{x}$=0-0=0.

    (e) False. There may be linearly dependent vectors in the spanning set. We do know that $\dim(V) \leq k$, however.

    (f) True. If A has a column or row of zeros, then its determinant is zero. Therefore, $\lambda = 0$ is a solution to $\det(A - \lambda I) = 0$.

(g) False. The projection of $\boldsymbol{y}$ onto $\boldsymbol{u}$ is a scalar multiple of $\boldsymbol{u}$.

(h) False. The number of distinct eigenvalues is not important, it is the number of linearly independent eigenvectors that is important. (Although, if A *has* 5 distinct eigenvalues, then we *can* guarantee that A is diagonalizable).

(i) True:

$$(y - \hat{y}) \cdot u = y \cdot u - \frac{y \cdot u}{u \cdot u} u \cdot u = 0$$

(j) True. Check it by computing

$$\begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_n^T \end{bmatrix} [\boldsymbol{u}_1 \ldots \boldsymbol{u}_n]$$

(k) True. Recall that A and B are similar if there is an invertible matrix P such that

$$A = PBP^{-1}$$

Therefore, $A - \lambda I = PBP^{-1} - \lambda I = PBP^{-1} - \lambda PP^{-1}$, so that

$$A - \lambda I = P(B - \lambda I)P^{-1}$$

so that the determinants on both sides are the same (use the fact that $\det(AB) = \det(A)\det(B)$).

(l) False. We said that $UU^T\boldsymbol{y}$ is the orthogonal projection of $\boldsymbol{y}$ onto the column space of U.

(m) False. Previously we said that similar matrices have similar eigenvalues, but the eigenvectors may change, because we'd solve:

$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}$$

which is the same as:

$$P(B - \lambda I)P^{-1}\boldsymbol{v} = \boldsymbol{0}$$

which is NOT necessarily the same as:

$$(B - \lambda I)\boldsymbol{v} = \boldsymbol{0}$$

(n) False. To get a basis for the column space of A, find which columns of A are linearly independent by identifying where the pivot columns are; go back to A to get the actual columns.

(o) False, unless they are both eigenvectors having the same eigenvalue (then we'd be talking about an eigenspace. Why? Let $\boldsymbol{u}, \boldsymbol{v}$ be eigenvectors of a matrix A. Then:

$$A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v} = \lambda_1 \boldsymbol{u} + \lambda_2 \boldsymbol{v}$$

which cannot necessarily be written as $\lambda(\boldsymbol{u} + \boldsymbol{v})$.

(p) False. Only one vector needs to be a linear combination of the others.

(q) False in general- If B is invertible, it is True. Recall that, more generally we said that the statement would be True, if the mapping $\boldsymbol{x} \to B\boldsymbol{x}$ is one to one.

(r) Nope. This is stating that $\boldsymbol{x} \to A\boldsymbol{x}$ is not onto.

(s) True. If A is invertible, then A^{-1} is invertible (that is, $(A^{-1})^{-1} = A$). If A^{-1} is invertible, its columns are linearly independent (by the Inverse Matrix Theorem).

(t) False. A and B might not be invertible separately.
We did say that if A and B are square, and AB is invertible, then A and B are invertible. (See Exercise 27, p. 124)

(u) True. If A is square with linearly independent columns, then it is invertible, and so is A^T (because $(A^T)^{-1} = (A^{-1})^T$). Since A^T is invertible, its columns are linearly independent (which are the rows of A).
Another version of the same thing: By the Invertible Matrix Theorem, A is invertible and its rows are linearly independent.

(v) False... Take out the addition signs, and it would be True.

28. If V has a finite basis, then any vector in V can be written as:

$$\boldsymbol{x} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n$$

where $c_1, \ldots, c_n$ are real numbers. Therefore, the isomorphism is:

$$\boldsymbol{x} \in V \quad \rightarrow \{c_1, \ldots, c_n\} \in \mathbb{R}^n$$