Review and Concept Relationships

Let T be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, expressed as $\mathbf{x} \to A\mathbf{x}$, where A is $m \times n$.

To say a linear mapping is:	Is Equivalent to:	
Onto	Range = Codomain	
	Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the cols of A	(I)
	$A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$	(II)
	The columns of A span $\mathbb{R}^m$	(III)
	"A pivot in every row"(*)	(IV)
One-to-one	$\boldsymbol{x}_1 \neq \boldsymbol{x}_2 \Rightarrow A\boldsymbol{x}_1 \neq A\boldsymbol{x}_2$	
	$A\boldsymbol{x} = \boldsymbol{b}$ has at most one solution	(V)
	$A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution	(VI)
	The columns of A are linearly independent.	(VII)
	"A pivot in every column"(*)	(VIII)

(*)- refers to the RREF of A

1. What is the definition of the span of a set of vectors?
2. What is the meaning behind "The set of vectors spans $\mathbb{R}^m$"?
3. Can a set of n vectors span $\mathbb{R}^m$ if $m > n$? If $m < n$?
4. What is the definition of linear independence in a set of vectors?
5. What is the heuristic definition of linear independence?
6. Can a set of n vectors in $\mathbb{R}^m$ be linearly independent if $m > n$? If $m < n$?
7. Show that (IV) implies (I), (II), (III).
8. Show that (VIII) implies (V), (VI), (VII).
9. In the case that A is square, why does any of (I)-(IV) imply that all 8 items are true? Does the same argument hold true if we only know that one of (V)-(VIII) hold?

   For the following two questions, T is the linear mapping, and the matrix A is the realization of T (so that $T(x) = A\mathbf{x}$)
10. If A is $m \times n$, $m > n$, Can T be $1-1$? Can T be onto?
11. If A is $m \times n$, $m < n$, Can T be $1-1$? Can T be onto?
12. Consider the statement: If AB is invertible, then B is invertible. (i) Show that this is FALSE in general (Come up with two matrices A, B). (ii) If A, B are $n \times n$, show that the statement is TRUE.
13. Show that, if T is a linear mapping and the only solution to $T(x) = 0$ is the trivial solution, then T must be $1-1$ (NOTE: Use arguments applicable to *functions*, not just matrices).
14. Show (by using the definition of linear maps) that, if T is invertible, then the inverse, S is also linear.