Complex Number Summary Sheet

You should read the Appendix in the Linear Algebra book (and in the Calculus book) for more details. This is meant to simply summarize the results.

1. Definition: If $z = a + bi$, then $\text{Re}(z) = a$, and $\text{Im}(a) = b$.

2. Definition: $\mathbb{C} = \left\{a + bi : \ a, b \in \mathbb{R}, \quad i = \sqrt{-1}\right\}$ =Space of complex numbers (it is a vector space).

3. The real numbers are a subspace of the complex numbers. (Set $b = 0$ in $a + bi$)- That is, $\mathbb{R} \subset \mathbb{C}$

4. Definition: $\mathbb{C}^n$ is the set of n-tuples of complex numbers, i.e., $\begin{bmatrix} a + bi \\ c + di \end{bmatrix} \in \mathbb{C}^2$

5. Operations on $\mathbb{C}$: Let $z = a + bi$, $w = c + di$.

   (a) $z + w = a + bi + c + di = (a + c) + (b + d)i$

   (b) $cz = ca + cbi$ (for $c \in \mathbb{R}$)

   (c) $zw = (a + bi)(c + di) = ac + adi + bci - bd = (ac - bd) + (ad + bc)i$

   (d) $\overline{z} = \overline{a + bi} = a - bi$ (Complex conjugation)

   (e) $|z| = \sqrt{z\overline{z}} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}$ =Modulus (or size) of z

   (f) $\dfrac{1}{z} = \dfrac{\overline{z}}{z\overline{z}} = \left(\dfrac{a}{a^2 + b^2}\right) + \left(\dfrac{b}{a^2 + b^2}\right) i$

6. Definition: The argument of z is the angle $\theta \in (-\pi, \pi]$ so that θ is the angle the vector $(a, b)^T$ makes with the vector $(1, 0)$ (translated to be between $-\pi$ and π). Formally,

$$\arg(z) = \begin{cases} \tan^{-1} \frac{b}{a} & \text{if} \quad a > 0, b \in \mathbb{R} \\ \frac{\pi}{2} & \text{if} \quad a = 0, b > 0 \\ -\frac{\pi}{2} & \text{if} \quad a = 0, b < 0 \\ \tan^{-1} \frac{b}{a} + \pi & \text{if} \quad a < 0, b > 0 \\ \tan^{-1} \frac{b}{a} - \pi & \text{if} \quad a < 0, b < 0 \end{cases}$$

   The argument is not defined for $z = 0$.

   In Matlab, the argument of $a + bi$ is: `atan2(b,a)`

7. Definition: (Euler's Formula) $e^{i\theta} = \cos(\theta) + i \sin(\theta)$

8. The polar form of $a + bi$ is: $re^{i\theta}$ again where $r = \sqrt{a^2 + b^2}$ and $\theta = \arg(z)$.

9. Multiplication in $\mathbb{C}$ by $z = a + bi$ corresponds to matrix multiplication in $\mathbb{R}^2$ by $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

   For example, if $w = c + di$ in $\mathbb{C}$, it corresponds to the vector $[c, d]^T \in \mathbb{R}^2$. Now, zw in $\mathbb{C}$ gives: $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$, which corresponds to the vector $[ac-bd, ad+bc]^T$. Alternatively in $\mathbb{R}^2$,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix}$$